

Logarithmic p -bases and arithmetical differential modules

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Abstract

We study several variants of the notion of (finite) log p -bases. We partially extend Berthelot's theory of arithmetic \mathcal{D} -modules in this context.

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Introduction

Berthelot's theory of arithmetic \mathcal{D} -modules in the context of varieties over a perfect field k of characteristic $p > 0$ is now well understood and is similar to l -adic étale cohomology, up to a certain point. For instance, we have the stability under six operations of the overholonomicity with Frobenius structures (see [CT12]) and a theory of weights (see [AC13]). Regarding nearby cycles or vanishing cycles however the situation is not clear yet since it requires to work over certain schemes which are not varieties over k (e.g. the spectrum of a henselian ring). In relation to the p -adic local monodromy theorem (see [And02], [Ked04], [Meb02]), Crew studied the case of $k[[t]]/k$ (e.g. he proved the holonomicity of F -isocrystals on the bounded Robba ring in [Cre06]). More recently, D. Pigeon extended the construction of Berthelot's sheaf of differential operators in a wider context, replacing étaleness by relative perfectness (note that $k[[t]]/k[t]$ is relatively perfect). It would also be useful (e.g. to be able to use Kedlaya's semistable reduction theorem ([Ked11])) to extend this construction further, by adding logarithmic structures as in C. Montagnon's thesis (see [Mon02]). This is the first goal of the present paper where we replace smoothness by log p -smoothness, a notion that relaxes and generalizes p -bases in the context of log schemes. Furthermore we obtain a partial extension of Berthelot's theory of arithmetic \mathcal{D} -modules. Over Laurent series fields, it would be interesting to compare our constructions with the theory of rigid cohomology as developed by C. Lazda and A. Pál (see [LP14a, LP14b, LP15]).

Let us describe the content of the paper. In the first chapter, we introduce the notions of log p -étaleness, log p -basis, log p -smoothness and log relative perfectness. We retrieve the usual local description of m -PD-envelopes (see [Ber96] or [Pig14]) in context of log p -smoothness. In the second chapter we extend Berthelot's theory of arithmetic \mathcal{D} -modules to log p -smooth fine log schemes. Namely, we define and study $\mathcal{D}_{X/S}^{(m)}$ (the sheaf of differential operators of level m) and we put a canonical right $\mathcal{D}_{X/S}^{(m)}$ -module structure on the dualizing sheaf $\omega_{X/S}$. We do this by brute force in order to avoid using a lacking suitable generalization of Grothendieck's extraordinary pullback as defined in [Har66].

In the third chapter we move to log p -adic formal schemes by taking limits as in [Ber96]. We conclude by defining extraordinary pull-backs, pushforwards and duals.

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Convention, notation of the paper

Let \mathcal{V} be a complete discrete valued ring of mixed characteristic $(0, p)$, π be a uniformizer, K its field of fractions, k its residue field which is supposed to be perfect. If $X \rightarrow Y$ is a morphism of log schemes, we denote by $\underline{f}: \underline{X} \rightarrow \underline{Y}$ the underlying morphism of schemes. An étale (resp. smooth) morphism of log schemes is a log étale and strict (resp. log smooth and strict) morphism. A fs log scheme is a fine saturated log scheme. The formal scheme $\mathrm{Spf} \mathcal{V}$ is meant for the p -adic topology. A formal \mathcal{V} -scheme means a p -adic formal scheme over $\mathrm{Spf} \mathcal{V}$.

Unless otherwise stated fiber products of fine log schemes (resp. fine formal log schemes) are always computed in the category of fine log schemes (resp. fine formal log schemes).

1 Log p -étaleness, log p -basis, log p -smoothness, relative log perfection

Let i be an integer and S be a fine log scheme over $\mathbb{Z}/p^{i+1}\mathbb{Z}$ and let (I_S, J_S, γ) be a quasi-coherent m -PD-ideal of \mathcal{O}_S .

1.1 Definitions

Notation 1.1.1. If X is a fine log scheme over \mathbb{F}_p , we denote by $F_X: X \rightarrow X$ the absolute Frobenius of X as defined by Kato in [Kat89, 4.7]. Unless otherwise specified, the log structure on $\mathbb{Z}/p^{i+1}\mathbb{Z}$ is the trivial one. If X is a log scheme over $\mathbb{Z}/p^{i+1}\mathbb{Z}$ and $0 \leq k \leq i$, we put $X_k := X \times_{\mathbb{Z}/p^{i+1}\mathbb{Z}} \mathbb{Z}/p^{k+1}\mathbb{Z}$.

Let \mathcal{I} be a quasi-coherent sheaf (for the Zariski topology) on a log scheme X . The preasheaf which associates $\mathcal{I}(U)^{(n)}$ (the subideal of $\mathcal{I}(U)$ generated by n th powers of elements of $\mathcal{I}(U)$) to an affine open set U of X is a quasi-coherent sheaf. We denote it by $\mathcal{I}^{(n)}$. Similarly, using [Ber00, A.1.5.(ii)], we get a quasi-coherent sheaf $\mathcal{I}^{\{n\}_{(m)}}$ such that for an affine open set U of X , $\mathcal{I}^{\{n\}_{(m)}}(U) = \mathcal{I}(U)^{\{n\}_{(m)}}$.

Definition 1.1.2. Let $u: Z \rightarrow X$ be a morphism of log-schemes.

1. We say that u is an immersion if \underline{u} is an immersion of schemes and if $u^*M_X \rightarrow M_Z$ is surjective (here u^* means the pullback of log structures [Kat89, 1.4]).
2. We say that u is a closed immersion if \underline{u} is a closed immersion of schemes and if $u^*M_X \rightarrow M_Z$ is surjective.
3. We say that u is an open immersion if \underline{u} is an open immersion of schemes and if $u^*M_X \rightarrow M_Z$ is an isomorphism.
4. Let n be an integer. A “log thickening of order (n) ” (resp. “log thickening of order n ”) is an *exact* closed immersion $u: U \hookrightarrow T$ such that $\mathcal{I}^{(n)} = 0$ (resp. such that $\mathcal{I}^n = 0$), where \mathcal{I} is the ideal of the closed immersion u .
5. A “ (p) -nilpotent log thickening” is a log thickening of order (p^a) for some integer a .

An S -immersion (resp. S -log thickening) is an immersion (resp. log thickening) which is an S -morphism.

Remark 1.1.3. 1. If $u: Z \rightarrow X$ and $f: X \rightarrow Y$ are two S -morphisms of log schemes such that $f \circ u$ is an S -immersion, then so is u .

2. We remark that we can decompose a (strict) S -immersion u into $u = u_1 \circ u_2$, where u_1 is an open S -immersion and u_2 is a (strict) closed S -immersion.
3. Let $u: U \hookrightarrow T$ be an S -log thickening of order (p^a) for some integer a . Since p is nilpotent in \mathcal{O}_T , by applying finitely many times the functor $\mathcal{I} \mapsto \mathcal{I}^{(p)}$ to the ideal defining u we obtain the zero ideal, which justifies the definition of (p) -nilpotent. This also implies that u is the composition of several S -log thickenings of order (p) .

Definition 1.1.4. 1. We denote by \mathcal{C} the category whose objects are S -immersions of fine log-schemes and whose morphisms $u' \rightarrow u$ are commutative diagrams of the form

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ u' \downarrow & & \downarrow u \\ Z' & \longrightarrow & Z. \end{array} \quad (1.1.4.1)$$

We say that $u' \rightarrow u$ is strict (resp. $u' \rightarrow u$ is flat cartesian) if f is strict (resp. f is strict, \underline{f} is flat and the square 1.2.1.1 is cartesian).

2. Let n be an integer. We denote by $\mathcal{C}_{(n)}$ (resp. \mathcal{C}_n , resp. $\mathcal{Thick}_{(p)}$) the full subcategory of \mathcal{C} whose objects are S -log thickenings of order (n) (resp. S -log thickening of order n , resp. (p) -nilpotent S -log thickenings).
3. We denote by \mathcal{C}^{sat} (resp. $\mathcal{C}_{(n)}^{\text{sat}}$, resp. $\mathcal{C}_n^{\text{sat}}$, resp. $\mathcal{Thick}_{(p)}^{\text{sat}}$) the full subcategory of \mathcal{C} (resp. $\mathcal{C}_{(n)}$, resp. \mathcal{C}_n , resp. $\mathcal{Thick}_{(p)}$) whose objects are morphisms of fs log-schemes.

Definition 1.1.5. Let $f: X \rightarrow Y$ be an S -morphism of fine log schemes.

1. We say that f is “log p -étale” (resp. “log p -unramified”) if it satisfies the following property: for any commutative diagram of fine log schemes of the form

$$\begin{array}{ccc} U & \xrightarrow{u_0} & X \\ \downarrow \iota & & \downarrow f \\ T & \xrightarrow{v} & Y \end{array} \quad (1.1.5.1)$$

such that ι is an object of $\mathcal{C}_{(p)}$, there exists a unique morphism (resp. there exists at most one morphism) $u: T \rightarrow X$ such that $u \circ \iota = u_0$ and $f \circ u = v$.

2. Replacing $\mathcal{C}_{(p)}$ by \mathcal{C}_2 we get the notion of “fine formally log étale” (resp. “fine formally log unramified”) morphism.
3. Replacing $\mathcal{C}_{(p)}$ by $\mathcal{C}_{(p)}^{\text{sat}}$ (resp. $\mathcal{C}_2^{\text{sat}}$), we get the notions of “fs log p -étale” and “fs log p -unramified” morphisms (resp. “fs formally log étale” and “fs formally log unramified” morphisms).
4. Replacing “fine log S -schemes” by “schemes” in the definition 1.1.5.1, we get the notion of p -étale (resp. p -unramified) morphism of schemes.

Lemma 1.1.6. Let $f: X \rightarrow Y$ be a strict S -morphism of fine log schemes, $\underline{f}: \underline{X} \rightarrow \underline{Y}$, $f^{\text{sat}}: X^{\text{sat}} \rightarrow Y^{\text{sat}}$, $\underline{f}^{\text{sat}}: \underline{X}^{\text{sat}} \rightarrow \underline{Y}^{\text{sat}}$ be the induced morphisms.

1. The morphism f is log p -étale if and only if \underline{f} is p -étale.
2. The morphism f is fs log p -étale if and only if $\underline{f}^{\text{sat}}$ is p -étale.

Proof. Left to the reader. □

Remark 1.1.7. With the last remark of 1.1.3 in mind, we can replace $\mathcal{C}_{(p)}$ by $\mathcal{Thick}_{(p)}$ (resp. $\mathcal{C}_{(p)}^{\text{sat}}$ by $\mathcal{Thick}_{(p)}^{\text{sat}}$) in the definition of log p -étale or log p -unramified (resp. fs log p -étale or fs log p -unramified).

Lemma 1.1.8. Let $f: X \rightarrow Y$ be an S -morphism of fine log-schemes and $f_0: X_0 \rightarrow Y_0$ be the induced S_0 -morphism. The morphism f is log p -étale (fs log p -étale) if and only if f is fine formally log étale (resp. fs formally log étale) and f_0 is log p -étale (resp. fs log p -étale). Similarly replacing everywhere “étale” by “unramified”.

Proof. Abstract nonsense. □

The following definition 1.1.9 will be extended in 1.1.13.

Definition 1.1.9. Let $f: X \rightarrow Y$ be an S_0 -morphism of fine log-schemes. We say that f is “fine log relatively perfect” (resp. “fs log relatively perfect”) if the diagram on the left (resp. on the right)

$$\begin{array}{ccc} X & \xrightarrow{F_X} & X \\ \downarrow f & & \downarrow f \\ Y & \xrightarrow{F_Y} & Y \end{array} , \quad \begin{array}{ccc} X^{\text{sat}} & \xrightarrow{F_{X^{\text{sat}}}} & X^{\text{sat}} \\ \downarrow f^{\text{sat}} & & \downarrow f^{\text{sat}} \\ Y^{\text{sat}} & \xrightarrow{F_{Y^{\text{sat}}}} & Y^{\text{sat}} \end{array} \quad (1.1.9.1)$$

is cartesian in the category of fine log-schemes (resp. fs log-schemes).

Lemma 1.1.10. Let $f: X \rightarrow Y$ be an S_0 -morphism of fine log-schemes. Then f is fs log relatively perfect (resp. fs log p -étale) if and only if so is f^{sat} .

Proof. The non respective case is obvious. The other one comes from 1.1.6. \square

Remark 1.1.11. Let $\iota: U \hookrightarrow T$ be a log S_0 -thickening of order (p) . Then the first projection $p_1: T \times_{F_T, T, \iota} U \rightarrow T$ is an isomorphism. Let $\varpi_\iota := p_2 \circ p_1^{-1}: T \rightarrow U$. We remark also that ϖ_ι is the unique morphism $T \rightarrow U$ making commutative the diagram

$$\begin{array}{ccc} U & \xrightarrow{F_U} & U \\ \downarrow \iota & \nearrow \varpi_\iota & \downarrow \iota \\ T & \xrightarrow{F_T} & T \end{array} \quad (1.1.11.1)$$

Lemma 1.1.12. Let $f: X \rightarrow Y$ be an S_0 -morphism of fine log-schemes. If f is fine log relatively perfect (resp. fs log relatively perfect) then f is log p -étale (resp. fs log p -étale).

Proof. Let us check the fine version. Let

$$\begin{array}{ccc} U & \xrightarrow{u_0} & X \\ \downarrow \iota & & \downarrow f \\ T & \xrightarrow{v} & Y \end{array} \quad (1.1.12.1)$$

be a commutative diagram of fs S_0 -log schemes such that i is log a S_0 -thickening of order (p) . First, let us check the unicity. Let $u: T \rightarrow X$ be a morphism such that $u \circ \iota = u_0$ and $f \circ u = v$. With the notation of the remark 1.1.11, we get $F_X \circ u = u \circ F_T = u \circ \iota \circ \varpi_\iota = u_0 \circ \varpi_\iota$. Since we have also $f \circ u = v$, we obtain the uniqueness of u from the cartesianity of 1.1.9.1. Now, let us check the existence. We have $F_Y \circ v = v \circ F_T = v \circ \iota \circ \varpi_\iota = f \circ u_0 \circ \varpi_\iota$. Hence, via the cartesianity of 1.1.9.1, we get the Y -morphism $u = (v, u_0 \circ \varpi_\iota): T \rightarrow X = Y \times_{F_Y, Y} X$. By definition, $f \circ u = v$. Moreover, $u \circ \iota = (v \circ \iota, u_0 \circ \varpi_\iota \circ \iota) = (f \circ u_0, u_0 \circ F_U) = (f \circ u_0, F_X \circ u_0) = u_0$.

Using the lemma 1.1.10, we reduce to check the second part for $f = f^{\text{sat}}$. Then we can proceed in the same way. \square

Definition 1.1.13. Let $f: X \rightarrow Y$ be an S -morphism of fine log-schemes. We say that f is “fine log relatively perfect” (resp. “fs log relatively perfect”) if f is fine formally log étale (resp. fs formally log étale) and if f_0 is fine log relatively perfect (resp. fs formally log étale).

Remark 1.1.14. 1. From lemma 1.1.12, this definition of log relative perfectness of 1.1.13 agrees that of 1.1.9 when $i = 0$, i.e. $S = S_0$.

2. Let $f: X \rightarrow Y$ be an S -morphism of fine log-schemes. From 1.1.10, f is fs log relatively perfect (resp. fs log p -étale) if and only if so is f^{sat} .

Proposition 1.1.15. A fine (resp. fs) log relatively perfect morphism is fine (resp. fs) log p -étale.

Proof. This is a consequence of 1.1.8 and 1.1.12. \square

Lemma 1.1.16. Let $f: X \rightarrow Y$ be a strict S -morphism of fine log schemes (resp. fs log schemes). Then f is fine log relatively perfect (resp. fs log relatively perfect) if and only if \underline{f} is relatively perfect as defined by Kato in [Kat91, 1.1].

Proof. The strict morphism f is fine (resp. fs) formally log étale if and only if \underline{f} is formally étale. Hence, we reduce to the case $i = 0$. In that case, this is clear from the fact that base change by a strict morphism commutes to the inclusion of the category of fine log schemes into that of all log schemes. \square

Lemma 1.1.17. 1. Let $f: X \rightarrow Y$ and $Y' \rightarrow Y$ be two S -morphisms of fine log-schemes. Set $X' := X \times_Y Y'$ in the category of fine log schemes and $f': X' \rightarrow Y'$ the projection. If f is fine log relatively perfect (resp. log p -étale, resp. fine formally log étale), then so is f' . The same holds verbatim with “fine” replaced by “fs”.

Proof. Since the fs part is similar, let us only consider the fine part. The respective cases are abstract nonsense. We reduce to check the non respective case. This is sufficient to treat the non respective case where $i = 0$. Since the outline of both diagrams

$$\begin{array}{ccccc} X' & \xrightarrow{F_{X'}} & X' & \longrightarrow & X \\ \downarrow f' & & \downarrow f' & \square & \downarrow f \\ Y' & \xrightarrow{F_{Y'}} & Y' & \longrightarrow & Y \end{array} \quad \begin{array}{ccccc} X' & \longrightarrow & X & \xrightarrow{F_X} & X \\ \downarrow f' & \square & \downarrow f & \square & \downarrow f \\ Y' & \longrightarrow & Y & \xrightarrow{F_Y} & Y \end{array} \quad (1.1.17.1)$$

are the same, we conclude. \square

Lemma 1.1.18. Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be two S -morphisms of fine log schemes. The morphisms f and g are fine log relatively perfect (resp. log p -étale, resp. fine formally log étale, resp. fs log relatively perfect, resp. fs log p -étale, resp. fs formally log étale) if and only if so are $f \circ g$ and g .

Proof. Abstract nonsense. \square

Proposition 1.1.19. Let $f: X \rightarrow Y$ be a log étale S -morphism of fine log-schemes. Then f is fs log relatively perfect.

Proof. From the last remark of 1.1.14, since f^{sat} is also log étale, we can suppose that $f = f^{\text{sat}}$. Since f is fs formally log étale, then we are reduced to the case $i = 0$. Next we observe that the case where f is strict follows from 1.1.16 and [Kat91, 1.2]. Using this case and 1.1.18, 1.1.17, we check that the relative perfectness of f is étale local on both X and Y . Using [Kat91, Theorem 3.5] (which give a étale local description of a log étale morphism), using properties 1.1.18 and 1.1.17, we are thus reduced to the case where $X = A_P$, $Y = A_Q$ and f is induced by a morphism $\phi: Q \rightarrow P$ of fs monoids such that the kernel and cokernel of ϕ^{gp} is finite of order prime to p . We have to prove that the left square below is cartesian in the category of fs log schemes:

$$\begin{array}{ccc} A_P & \xrightarrow{F_{A_P}} & A_P \\ \downarrow f & & \downarrow f \\ A_Q & \xrightarrow{F_{A_Q}} & A_Q \end{array} \quad \begin{array}{ccc} P & \xleftarrow{p} & P \\ \phi \uparrow & & \phi \uparrow \\ Q & \xleftarrow{p} & Q \end{array} \quad (1.1.19.1)$$

Since the functor $P \mapsto A_P$ from the category of fine saturated monoids to the category of fs log-schemes over $\mathbb{Z}/p^{i+1}\mathbb{Z}$ transforms cocartesian squares into cartesian squares, it is thus sufficient to check that the right square is cocartesian in the category of fine saturated monoids. Let us check that P satisfies the universal property. Let $\alpha: P \rightarrow O$ and $\beta: Q \rightarrow O$ be two morphisms of fine saturated monoids such that $\alpha \circ \phi = \beta \circ p$. Since O is saturated, if z is an element of O^{gp} such that pz is in O then z is an element of O . Hence, we reduce to check the universal property in the case where $P = P^{\text{gp}}$, $Q = Q^{\text{gp}}$ and $O = O^{\text{gp}}$. Let $x_0 \in P$. There exist $y \in Q$ and $x \in P$ such that $x_0 = \phi(y) + px$. (Indeed, let n be an integer prime to p such that $n \cdot \text{coker}(\phi) = 0$. There exists $y_0 \in Q$ such that $nx_0 = \phi(y_0)$. Using Bezout lemma, we get the desired result.) We define a morphism $h: P \rightarrow O$ satisfying $\alpha = h \circ p$ and $\beta = h \circ \phi$ by putting $h(x_0) := \beta(y) + \alpha(x)$. The morphism h is well defined. Indeed, if $x_0 = \phi(y) + px = \phi(y') + px'$, then $\phi(y - y') = p(x' - x)$ and then we can suppose $y' = 0$ and $x = 0$, i.e. $x_0 = \phi(y) = px'$. Since p is an isomorphism on $\text{coker}(\phi)$, there exists $z' \in Q$ such that $x' = \phi(z')$. Since p is an isomorphism on $\text{ker}(\phi)$, there exists $z'' \in \text{ker}(\phi)$ such that $y = p(z' + z'')$. We compute $\beta(y) = \beta(p(z' + z'')) = \alpha \circ \phi(z' + z'') = \alpha \circ \phi(z') = \alpha(x')$. The unicity of such h is also clear. \square

Remark 1.1.20. The “fine” version of proposition 1.1.19 is wrong, i.e. a log étale morphism is not necessarily fine log relatively perfect. Indeed, we have the following counter-example. Let $n \geq 2$, $p \nmid n$. In the category of fine (resp fine saturated) monoids the inductive limit of the diagram $\mathbb{N} \xleftarrow{p} \mathbb{N} \xrightarrow{n} \mathbb{N}$ is the submonoid of \mathbb{N} generated by p and n (resp. the associated saturated monoid, i.e. \mathbb{N} itself). Let $f: A_{\mathbb{N}} \rightarrow A_{\mathbb{N}}$ denote the morphism induced by $n: \mathbb{N} \rightarrow \mathbb{N}$. Then f is log étale but the left square of 1.1.9.1 is not cartesian in the category of fine log schemes.

Lemma 1.1.21. Let $f: X \rightarrow Y$ be an étale S -morphism of fine log-schemes. Then f is fine log relatively perfect.

Proof. Using 1.1.6 and 1.1.16, we reduce to check that an étale morphism of schemes is relatively perfect as defined by Kato in [Kat91, 1.1], which is well known. \square

Lemma 1.1.22. *Let $u: Z \hookrightarrow X$ be an object of $\mathcal{C}^{(m)}$ and \mathcal{J} be the ideal defining the exact closed immersion u . Let n be an integer prime to p . Then the homomorphism $1 + \mathcal{J} \rightarrow 1 + \mathcal{J}$ of groups (recall that $1 + \mathcal{J} \subset \mathcal{O}_X^*$) defined by $x \mapsto x^n$ is a bijection. Moreover, if \underline{X} is affine then for any $q > 0$, we have the vanishing $H^q(X, 1 + \mathcal{J}) = 0$.*

Proof. For N large enough, $\mathcal{J}^{(p^N)} = 0$. Using Bezout Lemma, we get the first assertion. Using [SGA4, VI.5.1], to check the second statement, we can suppose that \mathcal{J} is nilpotent. Hence, by devissage, we can reduce to the case where $\mathcal{J}^2 = 0$. Then $(1 + \mathcal{J}, \times)$ can be identified with $(\mathcal{J}, +)$ as a group. Since \mathcal{J} is quasi-coherent, we are down. \square

Proposition 1.1.23. *Let $f: X \rightarrow Y$ be a log étale S -morphism of fine log-schemes. Then f is log p -étale.*

Proof. With Lemma 1.1.21 and Proposition 1.1.15, if f is étale then f is log p -étale. Hence, we reduce to the case where $X = A_P$, $Y = A_Q$ and where there exists a chart of f subordinate to a morphism $\phi: Q \rightarrow P$ of fine monoids (see the definition [Ogu, II.2.1.7]) such that the kernel and cokernel of ϕ^{gp} is finite of order prime p (the characteristic of S). Using both properties of the Lemma 1.1.22, we can copy in the proof of Ogus [Ogu, IV.3.1.9] the part corresponding to the implication 3.1.9.1 \Rightarrow 3.1.9.2. \square

Remark 1.1.24. In the Lemma 1.1.22, we need that p is nilpotent in \mathcal{O}_X . But without using Lemma 1.1.22 we can check that the proposition is still valid if we replace S by any log scheme as follows. Using Lemma 1.1.21 and proposition 1.1.15, the case where f is strict is already known. So we are reduced to the case where $X = A_P$, $Y = A_Q$ and where there exists a chart of f subordinate to a morphism of fine monoids of the form $\phi: Q \rightarrow P$ such that the kernel and cokernel of ϕ^{gp} is finite of order prime to the characteristic of S . Let N be the product of the cardinal of the kernel and cokernel of ϕ^{gp} . Using base change stability property, we can suppose S of the form $\text{Spec } \mathbb{Z}[\frac{1}{N}]$. In particular, we can suppose that S is noetherian. Let a diagram be as in 1.1.5.1 with our f . We can suppose that v is subordinate to a morphism of finitely generated monoids of the form $Q \rightarrow Q_T$ and that \underline{T} is affine (see [Ogu, II.2.2.3]). Replacing A_P by $A_P \times_{A_Q} A_{Q_T}$, we can suppose that $Q \rightarrow Q_T$ is the identity. Let T' and U' defined so that $\mathcal{O}_{T'}$ and $\mathcal{O}_{U'}$ are respectively the image of $\mathcal{O}_S[Q]$ in \mathcal{O}_T and \mathcal{O}_U and so that the morphisms $T' \hookrightarrow A_Q$ and $U' \hookrightarrow T'$ (whose underlying morphism of schemes are closed immersions) are exact. Then since T' is noetherian, the immersion $U' \hookrightarrow T'$ is nilpotent. Since f is log étale, we get the desired unique factorization property.

1.1.25. The following diagram summarizes the relations found until now.

$$\begin{array}{ccccccc}
 & & \xrightarrow{\quad 1.1.23 \quad} & & & & \\
 \text{log étale} & \xrightarrow{1.1.20} & \text{fine log relatively perfect} & \xrightarrow{1.1.15} & \text{log } p\text{-étale} & \Longrightarrow & \text{fine formally log étale} \\
 & \searrow 1.1.19 & \Downarrow & & \Downarrow & & \Downarrow \\
 & & \text{fs log relatively perfect} & \xrightarrow{1.1.15} & \text{fs log } p\text{-étale} & \Longrightarrow & \text{fs formally log étale}
 \end{array} \tag{1.1.25.1}$$

From now on we will work with fine (not necessarily saturated) log schemes. For our purpose the most relevant notion among those of the above diagram thus turns out to be log p -étaleness. The reader which is only interested in the category of fs log schemes should replace log p -étaleness by fs log p -étaleness in the sequel.

Definition 1.1.26. Let $f: X \rightarrow Y$ be an S -morphism of fine log-schemes.

1. We say that a finite set $(b_\lambda)_{\lambda=1, \dots, r}$ of elements of $\Gamma(X, M_X)$ is a “finite log p -basis” if the corresponding Y -morphism $X \rightarrow Y \times_{\mathbb{Z}} A_{\mathbb{N}^r}$ is log p -étale.
2. We say that f is “log p -smooth” if, étale locally on X and Y , f has a finite log p -basis.
3. We say that f is “ p -étale” (resp. “ p -smooth”) if it is strict and log p -étale (resp. log p -smooth).

1.1.27. The collection of log p -smooth (resp. log p -étale) S -morphisms of fine log-schemes is stable under base change and under composition.

Definition 1.1.28. Let $f: X \rightarrow Y$ be an S -morphism of fine log-schemes.

1. We say that a finite set $(t_\lambda)_{\lambda=1,\dots,r}$ of elements of $\Gamma(X, \mathcal{O}_X)$ are “log p -étale coordinates” if the corresponding Y -morphism $X \rightarrow Y \times_{\mathbb{Z}} \mathbb{A}^r$, where \mathbb{A}^r is the r th affine space over \mathbb{Z} endowed with the trivial logarithmic structure, is log p -étale.
2. We say that a finite set $(t_\lambda)_{\lambda=1,\dots,r}$ of elements of $\Gamma(X, \mathcal{O}_X)$ is a “finite p -basis” if they are log p -étale coordinates and if f is strict (this is equivalent to say that the Y -morphism $X \rightarrow Y \times_{\mathbb{Z}} \mathbb{A}^r$ is p -étale).

Remark 1.1.29. The notion of log p -étale coordinates is not well adapted to the context of log schemes (for instance, $A_{\mathbb{N}, \mathbb{Z}/p^{i+1}\mathbb{Z}}$ has no log p -étale coordinates over $\text{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})$, even étale locally). In the case of a strict morphism however it will allow to compare the logarithmic situation with the non logarithmic one (e.g. see 2.2.3).

Remark 1.1.30. Let $f: X \rightarrow Y$ be an S -morphism of fine log-schemes.

1. Let $(t_\lambda)_{\lambda=1,\dots,r}$ in $\Gamma(X, \mathcal{O}_X^*)$. Then $(t_\lambda)_{\lambda=1,\dots,r}$ form a log p -basis if and only if they are log p -étale coordinates. Indeed, in both cases, the elements $(t_\lambda)_{\lambda=1,\dots,r}$ induce a morphism of the form $X \rightarrow Y \times_{\mathbb{Z}} \mathbb{G}_m^r$. We conclude by using 1.1.18.
2. Suppose that f is strict. If the morphism f has a finite p -basis or log p -basis with r elements then, étale locally on X , there exist elements $(t_\lambda)_{\lambda=1,\dots,r}$ of $\Gamma(X, \mathcal{O}_X^*)$ which are a p -basis of f .
3. Suppose that f is strict. The morphism f is p -smooth if and only if, étale locally on X and Y , f has a finite p -basis.

1.2 m -PD-envelopes

Let us fix some definitions.

Definition 1.2.1. 1. Let $\mathcal{C}^{(m)}$ (resp. $\mathcal{C}_n^{(m)}$) be the category whose objects are pairs (u, δ) where u is an exact closed S -immersion $Z \hookrightarrow X$ of fine log-schemes over S and δ is an m -PD-structure on the ideal \mathcal{I} of \mathcal{O}_X defining u which is compatible (see definition [Ber96, 1.3.2.(ii)]) with γ (resp. and such that $\mathcal{I}^{\{n+1\}(m)} = 0$) and whose morphisms $(u', \delta') \rightarrow (u, \delta)$ are commutative diagrams of the form

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ u' \uparrow & & \uparrow u \\ Z' & \longrightarrow & Z \end{array} \quad (1.2.1.1)$$

such that m -PD-structures δ and δ' are compatible (via f).

We say that $(u', \delta') \rightarrow (u, \delta)$ is strict (resp. $(u', \delta') \rightarrow (u, \delta)$ is flat cartesian) if f is strict (resp. f is strict, \underline{f} is flat, the square 1.2.1.1 is cartesian and δ' is the m -PD-structure induced by δ ; recall that this latter structure exists from [Ber96, 1.4.6]).

2. Let u be an object of \mathcal{C} (see the notation 1.1.4). An “ m -PD-envelope compatible with γ of u ” is an object (u', δ') of $\mathcal{C}^{(m)}$ endowed with a morphism $u' \rightarrow u$ in \mathcal{C} satisfying the universal property: for any object (u'', δ'') of $\mathcal{C}^{(m)}$ endowed with a morphism $f: u'' \rightarrow u$ of \mathcal{C} there exists a unique morphism $(u'', \delta'') \rightarrow (u', \delta')$ of $\mathcal{C}^{(m)}$ such that the composition of $u'' \rightarrow u'$ with $u' \rightarrow u$ is f . The unicity up to canonical isomorphism of the m -PD-envelope compatible with γ of u is obvious. The existence is checked below (see 1.2.11).
3. We denote by $For^{(m)}: \mathcal{C}^{(m)} \rightarrow \mathcal{C}$ (resp. $For_n^{(m)}: \mathcal{C}_n^{(m)} \rightarrow \mathcal{C}$) the canonical functor.

Remark 1.2.2. Let (u, δ) be an object of $\mathcal{C}^{(m)}$ (or $\mathcal{C}_n^{(m)}$). Then u is an object of $\mathcal{T}hick_{(p)}$ (see the notation 1.1.2).

Remark 1.2.3. The category \mathcal{C} has fibered products (but this is less clear for $\mathcal{C}^{(m)}$). More precisely, let $u: Z \hookrightarrow X$, $u': Z' \hookrightarrow X'$, $u'': Z'' \hookrightarrow X''$ be some objects of \mathcal{C} ; let $u' \rightarrow u$ and $u'' \rightarrow u$ be two morphisms of \mathcal{C} . Then $u' \times_u u''$ is the immersion $Z' \times_Z Z'' \hookrightarrow X' \times_X X''$.

1.2.4. Forgetting log structures, i.e. replacing in 1.1.4 and 1.2.1 fine log-schemes by schemes, we define similarly the categories $\underline{\mathcal{C}}$, $\underline{\mathcal{C}}^{(m)}$ and $\underline{\mathcal{C}}_n^{(m)}$. Following Berthelot (see the second remark after the proposition [Ber96, 1.4.6]), the forgetful functor $\underline{For}^{(m)}: \underline{\mathcal{C}}^{(m)} \rightarrow \underline{\mathcal{C}}$ defined by $(\underline{u}, \delta) \mapsto \underline{u}$ has a right adjoint that we will denote by $\underline{P}_{(m), \gamma}$. If \underline{u} is an object of $\underline{\mathcal{C}}$ then $\underline{P}_{(m), \gamma}(\underline{u})$ is called the “ m -PD-envelope compatible with γ of \underline{u} ”. Moreover, the morphism of schemes induced by the targets of $\underline{P}_{(m), \gamma}(\underline{u}) \rightarrow \underline{u}$ is affine (see [Ber96]).

1.2.5. Let $u: Z \hookrightarrow X$ be an exact S -immersion of fine log-schemes. Set $(\underline{v}, \delta) := \underline{P}_{(m), \gamma}(\underline{u})$ (see 1.2.4). Let (v, δ) be the object of $\mathcal{C}^{(m)}$ whose underlying object of $\underline{\mathcal{C}}^{(m)}$ is (\underline{v}, δ) and v is defined so that the morphism $v \rightarrow u$ of \mathcal{C} is strict (see the definition 1.1.4). Then (v, δ) is the m -PD-envelope compatible with γ of u .

1.2.6. Let $u: Z \hookrightarrow X$ be an S -immersion of fine log-schemes. Let \bar{z} be a geometric point of Z . Using [Kat89, 4.10.1] and [SGA1, I.8.1] (we notice that noetherian hypotheses are not necessary), we check that there exists a commutative diagram of the form

$$\begin{array}{ccccc} \tilde{X} & \xrightarrow{f} & X' & \xrightarrow{g} & X \\ & \searrow v' & \uparrow u' & \square & \uparrow u \\ & & Z' & \xrightarrow{h} & Z \end{array}$$

such that the square is cartesian, f is log étale, \underline{f} is affine, g is étale, v' is an exact closed S -immersion and h is an étale neighborhood of \bar{z} in Z .

Remark 1.2.7. Let $\alpha: (u', \delta') \rightarrow (u, \delta)$ be a flat cartesian morphism of $\mathcal{C}^{(m)}$. Let (u'', δ'') be an object of $\mathcal{C}^{(m)}$ and $\beta: u'' \rightarrow u'$ be a morphism of \mathcal{C} . We remark that if $For^{(m)}(\alpha) \circ \beta$ is in the image of $For^{(m)}$ then so is β (use [Ber96, 1.4.6]).

1.2.8. Let $u' \rightarrow u$ be a flat cartesian morphism of \mathcal{C} , i.e. let

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ u' \uparrow & \square & \uparrow u \\ Z' & \longrightarrow & Z \end{array}$$

be a cartesian square whose morphism g is strict and \underline{g} is flat. Suppose that the m -PD-envelope compatible with γ of u exists (in fact, this existence will be proved later in 1.2.11). Let (v, δ) be this m -PD-envelope. Let $(v', \delta') := (v \times_u u', \delta')$ where δ' is the unique m -PD-structure so that $(v \times_u u', \delta') \rightarrow (v, \delta)$ is flat cartesian (the existence of the extension δ' comes from [Ber96, 1.4.6]). With the remark 1.2.7, we check that (v', δ') is an m -PD-envelope compatible with γ of u' .

Lemma 1.2.9. Let $f: X \rightarrow Y$ be a log p -étale S -morphism, $u: Z \hookrightarrow X$ and $v: Z \hookrightarrow Y$ be two S -immersions of fine log schemes such that $v = f \circ u$. If the m -PD envelope compatible with γ of u exists then it is also an m -PD envelope of v .

Proof. Let $P(u)$ be the m -PD envelope compatible with γ of u . We check that the composition of the canonical morphism $P(u) \rightarrow u$ with the morphism $u \rightarrow v$ (induced by f) satisfies the universal property of the m -PD envelope compatible with γ of v , which gives the desired result. Indeed, if (v', δ') is an object of $\mathcal{C}^{(m)}$ and $g: v' \rightarrow v$ is a morphism of \mathcal{C} , then using the universal property of p -étaleness of 1.1.5 we get a unique morphism $v' \rightarrow u$ of \mathcal{C} whose composition with $u \rightarrow v$ gives g . We conclude using the universal property of $P(u)$. \square

Lemma 1.2.10. The canonical functor $For_n: \mathcal{C}_n^{(m)} \rightarrow \mathcal{C}^{(m)}$ has a right adjoint. We denote by $P^n: \mathcal{C}^{(m)} \rightarrow \mathcal{C}_n^{(m)}$ this right adjoint functor.

Proof. It follows from [Ber96, 1.3.8.(iii)] that if u is an object of $\mathcal{C}^{(m)}$ and \mathcal{I} is the ideal defining the exact closed immersion u , then $P^n(u)$ is the exact closed immersion corresponding to $\mathcal{I}^{\{n+1\}_{(m)}}$. \square

Proposition 1.2.11. Let $u: Z \hookrightarrow X$ be an object of \mathcal{C} .

1. The m -PD-envelope compatible with γ of u exists. In other words, the canonical functor $For^{(m)}: \mathcal{C}^{(m)} \rightarrow \mathcal{C}$ has a right adjoint. We denote by $P_{(m), \gamma}: \mathcal{C} \rightarrow \mathcal{C}^{(m)}$ this right adjoint functor. Similarly, we get the right adjoint functor $P_{(m), \gamma}^n: \mathcal{C} \rightarrow \mathcal{C}_n^{(m)}$ of the canonical functor $For_n^{(m)}: \mathcal{C}_n^{(m)} \rightarrow \mathcal{C}$.
2. If γ extends to Z then the source of $P_{(m), \gamma}(u)$ is Z .
3. By denoting abusively by $P_{(m), \gamma}(u)$ (resp. $P_{(m), \gamma}^n(u)$) the target of the arrow $P_{(m), \gamma}(u)$ (resp. $P_{(m), \gamma}^n(u)$), the underlying morphism of schemes of $P_{(m), \gamma}(u) \rightarrow X$ (resp. $P_{(m), \gamma}^n(u) \rightarrow X$) is affine. We denote by $\mathcal{P}_{(m), \gamma}(u)$ (resp. $\mathcal{P}_{(m), \gamma}^n(u)$) the quasi-coherent \mathcal{O}_X -algebra so that $\underline{P_{(m), \gamma}(u)} = \text{Spec}(\mathcal{P}_{(m), \gamma}(u))$ (resp. $\underline{P_{(m), \gamma}^n(u)} = \text{Spec}(\mathcal{P}_{(m), \gamma}^n(u))$). The m -PD structure of $\mathcal{P}_{(m), \gamma}(u)$ will be denoted by $(\mathcal{I}_{(m), \gamma}(u), \mathcal{J}_{(m), \gamma}(u), [1])$

Proof. From Lemma 1.2.10, if the right adjoint $P_{(m),\gamma}$ exists then $P_{(m),\gamma}^n = P^n \circ P_{(m),\gamma}$ exists. Hence, we come down to check the existence of $P_{(m),\gamma}$. Since the proposition is étale local in X (see the definition [SGA3, IV.6.3] of the étale topology and the proposition [SGA3, IV.6.3.1(iv)] for an alternative definition), by 1.2.6, we may thus assume that there exists a commutative diagram of the form

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{f} & X \\ & \searrow \tilde{u} & \uparrow u \\ & & Z \end{array}$$

such that f is log étale, \underline{f} is affine and \tilde{u} is an exact closed S -immersion. In that case, following 1.2.5 the m -PD-envelope compatible with γ of \tilde{u} exists and the induced object of $\mathcal{C}^{(m)}$ is $\underline{P}_{(m),\gamma}(\tilde{u})$. Following 1.2.9, the m -PD-envelope compatible with γ of u exists and is isomorphic to that of \tilde{u} . Concerning the second statement, when γ extends to Z , following [Ber96], the source of the immersion $\underline{P}_{(m),\gamma}(\tilde{u})$ is \underline{Z} . Since $\underline{P}_{(m),\gamma}(\tilde{u})$, \tilde{u} and $\underline{P}_{(m),\gamma}(\tilde{u}) \rightarrow \tilde{X}$ are exact, we conclude. We check the third statement recalling that $\underline{P}_{(m),\gamma}(\tilde{u})$ is affine over \tilde{X} (see 1.2.4). \square

1.2.12 (The case of an exact closed immersion). Let $u: Z \hookrightarrow X$ be an exact closed S -immersion of fine log-schemes and $\mathcal{I}(u)$ be the ideal defining u . We denote by $u^{(m)}: Z^{(m)} \hookrightarrow X$ the exact closed S -immersion of fine log-schemes so that $\mathcal{I}(u)^{(p^m)}$ is the ideal defining $u^{(m)}$. Since the closed immersion is exact, in the proof of 1.2.11, we can skip the part concerning the exactification. Hence, we remark that, as in the proof of [Ber96, 1.4.1], we get the equality

$$P_{(m),\gamma}(u) = P_{(0),\gamma}(u^{(m)}). \quad (1.2.12.1)$$

We have also the same construction as in the the proof of [Ber96, 1.4.1] (too long to be given here in few words) of the m -PD ideal $(\mathcal{I}_{(m),\gamma}(u), \mathcal{I}_{(m),\gamma}(u)^{[1]})$ of $\mathcal{P}_{(m),\gamma}(u)$ directly from the level 0 case. For the detailed descriptions, see the proof of [Ber96, 1.4.1]. These descriptions, in particular 1.2.12.1, are useful to check the Frobenius descent for arithmetic \mathcal{D} -modules (see [Ber00]).

Proposition 1.2.13. *For any integer n , the canonical functor $\mathcal{C}_n \rightarrow \mathcal{C}$ (resp. $\mathcal{C}_{(p^n)} \rightarrow \mathcal{C}$) has a right adjoint functor which we will denote by $P^n: \mathcal{C} \rightarrow \mathcal{C}_n$ (resp. $P^{(p^n)}: \mathcal{C} \rightarrow \mathcal{C}_{(p^n)}$). Let $u: Z \hookrightarrow X$ be an object of \mathcal{C} . Then Z is also the source of $P^n(u)$ and $P^{(p^n)}(u)$.*

Proof. For the exactification, we proceed as in the beginning of the proof of 1.2.11. The rest of the proof is obvious. \square

Proposition 1.2.14. *Let $f: X \rightarrow Y$ be an S -morphism of fine log schemes and $\Delta_{X/Y}: X \hookrightarrow X \times_Y X$ (as always the product is taken in the category of fine log schemes) be the diagonal S -immersion. The morphism f is fine formally log unramified (resp. log p -unramified) if and only if $P^2(\Delta_{X/Y})$ (resp. $P^{(p)}(\Delta_{X/Y})$) is an isomorphism.*

Proof. Abstract non sense. \square

Lemma 1.2.15. *Let $u \rightarrow v$ be a morphism of \mathcal{C} . Let δ be the m -PD-structure of $P_{(m),\gamma}(v)$ and $w := P_{(m),\gamma}(v) \times_v u$ (this is the product in \mathcal{C}). Then $P_{(m),\delta}(w)$ and $P_{(m),\gamma}(u)$ are isomorphic in $\mathcal{C}^{(m)}$.*

Proof. We have by composition the morphism $P_{(m),\delta}(w) \rightarrow w \rightarrow u$. We check easily that $P_{(m),\delta}(w) \rightarrow u$ satisfies the universal property of $P_{(m),\gamma}(u) \rightarrow u$. \square

Proposition 1.2.16. *Let $f: X \rightarrow Y$ be an S -morphism of fine log-schemes endowed with a log p -basis $(b_\lambda)_{\lambda=1,\dots,r}$. Let $u: Z \hookrightarrow X$ and $v: Z \hookrightarrow Y$ be two S -immersions of fine log schemes such that $v = f \circ u$. Suppose given $y_\lambda \in \Gamma(Y, M_Y)$ whose images in $\Gamma(Z, M_Z)$ coincide with the images of b_λ (such y_λ 's always exist étale locally on Y). Let $P(u): T' \hookrightarrow D'$ and $P(v): T \hookrightarrow D$ be the m -PD envelopes compatible with γ of u and v respectively. Let u_λ be the unique section of $\ker(\mathcal{O}_{D'}^* \rightarrow \mathcal{O}_{T'}^*)$ such that $b_\lambda = f^*(y_\lambda)u_\lambda$ in $M_{D'}$ (with the multiplicative notation). Then, we have the isomorphism of m -PD- \mathcal{O}_D -algebras*

$$\begin{aligned} \mathcal{O}_D < T_1, \dots, T_r >_{(m)} &\xrightarrow{\sim} \mathcal{O}_{D'} \\ T_\lambda &\mapsto u_\lambda - 1. \end{aligned} \quad (1.2.16.1)$$

Proof. Using lemma 1.2.9 (and the first remark of 1.1.3), we may assume that $X = Y \times_{\mathbb{Z}} A_{\mathbb{N}^r}$ and that the family $(b_\lambda)_{\lambda=1,\dots,r}$ are the elements of $\Gamma(X, M_X)$ corresponding to the canonical basis $(e_\lambda)_{\lambda=1,\dots,r}$ of \mathbb{N}^r . Using lemma 1.2.15, we may furthermore assume that $Y = S$, I_S is the ideal of $Z \hookrightarrow Y$ (ie. $D = Y$) and that γ is the canonical m -PD structure of D .

Let \bar{z} be a geometric point of Z . From 1.2.6, there exists a commutative diagram of the form

$$\begin{array}{ccccc} U & \xrightarrow{g} & X = Y \times_{\mathbb{Z}} A_{\mathbb{N}^r} & \xrightarrow{f} & Y \\ \uparrow w & & \uparrow u & \nearrow v & \\ W & \xrightarrow{h} & Z & & \end{array}$$

where f is the first projection, g is log étale, h is an étale neighborhood of \bar{z} in Z and w is an exact closed S -immersion. We denote by c_λ the elements of $\Gamma(U, M_U)$ defining g and we set $v_\lambda := \frac{c_\lambda}{(f \circ g)^*(y_\lambda)} \in \text{Ker}(\Gamma(U, M_U^{\text{gp}}) \rightarrow \Gamma(W, M_W^{\text{gp}}))$.

1) In this step, we reduce by étale descent to the case where $h = \text{id}$. Since w is an exact closed immersion, we have $(M_U/\mathcal{O}_U^*)_{\bar{z}} = (M_W/\mathcal{O}_W^*)_{\bar{z}}$ (use [Kat89, 1.4.1]) and thus $(M_U^{\text{gp}}/\mathcal{O}_U^*)_{\bar{z}} = (M_W^{\text{gp}}/\mathcal{O}_W^*)_{\bar{z}}$. Shrinking U if necessary we may thus assume that $v_\lambda \in \Gamma(U, \mathcal{O}_U^*)$. According to [SGA1, I.8.1] (we notice that noetherian hypotheses are not necessary), there exist an étale neighborhood $Y' \rightarrow Y$ of \bar{z} in Y and an open immersion $\iota: Z' := Z \times_Y Y' \rightarrow W$ which is a morphism of étale neighborhoods of \bar{z} in Z . Let us use a prime to denote the base change by $Y' \rightarrow Y$ of a Y -log scheme or a morphism of Y -log schemes. Using the open Y -immersion $j := (\iota, v'): Z' \rightarrow W'$ (see the definition 1.1.2) we get a commutative diagram over Y

$$\begin{array}{ccc} U' & \xrightarrow{g'} & X' = Y' \times_{\mathbb{Z}} A_{\mathbb{N}^r} \\ & \nwarrow w' \circ j & \uparrow u' \\ & & Z' \end{array}$$

Using 1.1.3.2 we may assume (shrinking U if necessary) that the exact Y -immersion $w' \circ j$ is closed. Since the proposition is étale local on Y , we can drop the primes, i.e. we can suppose $h = \text{id}$.

2) From the step 1), we have to prove that the m -PD envelope D' of w (and thus of u by 1.2.9) is such that the m -PD polynomial \mathcal{O}_Y -algebra in T_1, \dots, T_r is isomorphic to $\mathcal{O}_{D'}$ (the isomorphism is given by $T_\lambda \mapsto v_\lambda - 1 = u_\lambda - 1$ where the latter elements means their image in $\mathcal{O}_{D'}$). Consider the Y -morphism $\phi: U \rightarrow Y \times_{\mathbb{Z}} A_{\mathbb{N}^r}$ defined by the v_λ 's. Since the $d \log c_\lambda$'s form a basis of $\Omega_{U/Y}$ (because g is log étale), then so do the $d \log v_\lambda$'s. This implies that the canonical map $\phi^* \Omega_{Y \times_{\mathbb{Z}} A_{\mathbb{N}^r}} \rightarrow \Omega_{U/Y}$ induced by ϕ is an isomorphism. Since U/Y is log smooth we get that ϕ is log smooth (use [Ogu, IV.3.2.3.2]) and then log étale. We have a commutative diagram over Y as follows:

$$\begin{array}{ccc} U & \xrightarrow{\phi} & Y \times_{\mathbb{Z}} A_{\mathbb{N}^r} \\ & \nwarrow w & \uparrow \iota \circ v \\ & & Z \end{array}$$

where $\iota: Y \rightarrow Y \times_{\mathbb{Z}} A_{\mathbb{N}^r}$ is the Y -morphism defined by $e_\lambda \mapsto 1 \in \Gamma(Y, M_Y)$. We are thus reduced to compute the m -PD-envelope compatible with γ of the exact closed Y -immersion $\iota \circ v$. By using the remark [Ber96, 1.4.3.(iii)] and 1.2.5, we can suppose that $v = \text{id}$, i.e. that the m -PD ideal given by v is 0. The Y -immersion ι is equal to the composition of the natural log étale morphism $Y \times_{\mathbb{Z}} A_{\mathbb{Z}^r} \rightarrow Y \times_{\mathbb{Z}} A_{\mathbb{N}^r}$ with the exact closed Y -immersion $i: Y \rightarrow Y \times_{\mathbb{Z}} A_{\mathbb{Z}^r}$ defined by $e_\lambda \mapsto 1 \in \Gamma(Y, M_Y)$. Let x_λ be the element of $\Gamma(Y \times_{\mathbb{Z}} A_{\mathbb{Z}^r}, \mathcal{O}_{Y \times_{\mathbb{Z}} A_{\mathbb{Z}^r}})$ induced by e_λ . Since the ideal of the exact closed immersion i is generated by the regular sequence $(x_\lambda - 1)_{\lambda=1,\dots,r}$, using [Ber96, 1.5.3] and 1.2.5 we check that the morphism $\mathcal{O}_Y < T_1, \dots, T_r >_{(m)} \rightarrow \mathcal{O}_{P_{(m),\gamma}(\iota)} = \mathcal{O}_{P_{(m),\gamma}(i)}$ given by $T_\lambda \mapsto x_\lambda - 1$ is an isomorphism. \square

Notation 1.2.17. With the notation of proposition 1.2.16, we denote by $\mathcal{O}_D < T_1, \dots, T_r >_{(m),n}$ the n -truncated m -PD-polynomial ring. We get the isomorphism $\mathcal{O}_D < T_1, \dots, T_r >_{(m),n} \xrightarrow{\sim} \mathcal{P}_{(m),\gamma}^n(u)$ of m -PD- \mathcal{O}_D -algebra.

Proposition 1.2.18. Let $f: X \rightarrow Y$ be an S -morphism of fine log-schemes endowed with log p -étale coordinates $(t_\lambda)_{\lambda=1,\dots,r}$. Let $u: Z \hookrightarrow X$ and $v: Z \hookrightarrow Y$ be two S -immersions of fine log schemes such that $v = f \circ u$. Suppose that there exist $y_\lambda \in \Gamma(Y, \mathcal{O}_Y)$ whose images in $\Gamma(Z, \mathcal{O}_Z)$ coincide with the images of t_λ . Let $P(u): T' \hookrightarrow D'$ and

$P(v): T \hookrightarrow D$ be the m -PD envelopes compatible with γ of u and v respectively. Then, we have the isomorphism of m -PD- \mathcal{O}_D -algebras

$$\begin{aligned} \mathcal{O}_D < T_1, \dots, T_r >_{(m)} &\xrightarrow{\sim} \mathcal{O}_{D'} \\ T_\lambda &\mapsto t_\lambda - f^*(y_\lambda), \end{aligned} \quad (1.2.18.1)$$

where t_λ and $f^*(y_\lambda)$ denote the corresponding element of $\mathcal{O}_{D'}$ induced by $D' \rightarrow X$.

Proof. This is similar to 1.2.16 and even easier (for instance, since $\mathbb{A}^r \times Y \rightarrow Y$ is strict, then the part concerning the exactification of u is useless, i.e. with the notation of the proof of 1.2.16 we can simply take $w = u$). \square

2 Differential operators of level m over log p -smooth log schemes

Let S be a fine log scheme over $\mathbb{Z}/p^{i+1}\mathbb{Z}$ and let (I_S, J_S, γ) be a quasi-coherent m -PD-ideal of \mathcal{O}_S . Let $f: X \rightarrow S$ be a log p -smooth morphism of fine log-schemes such that γ extends to X (e.g. from [Ber96, 1.3.2.c]) when $J_S + p\mathcal{O}_S$ is locally principal).

2.1 Sheaf of principal parts of level m

2.1.1. Let $\Delta: X \rightarrow X \times_S X$ be the diagonal morphism, $\Delta_{X/S, (m)} := P_{(m), \gamma}(\Delta)$, $\Delta_{X/S, (m)}^n := P_{(m), \gamma}^n(\Delta)$, $\mathcal{P}_{X/S, (m)}^n := \mathcal{P}_{(m), \gamma}^n(\Delta)$. We denote by $M_{X/S, (m)}$ (resp. $M_{X/S, (m)}^n$) the log structure of $\Delta_{X/S, (m)}$ (resp. $\Delta_{X/S, (m)}^n$). We denote abusively the target of $\Delta_{X/S, (m)}$ by $\Delta_{X/S, (m)}$. Since γ extends to X , the source of $\Delta_{X/S, (m)}^n$ is X , i.e. $\Delta_{X/S, (m)}^n$ is a closed immersion of the form $X \hookrightarrow (\text{Spec } \mathcal{P}_{X/S, (m)}^n, M_{X/S, (m)}^n)$.

Let $p_1, p_0: \Delta_{X/S, (m)} \rightarrow X$ be respectively the composition of the canonical morphism $\Delta_{X/S, (m)} \rightarrow X \times_S X$ with the right and left projection $X \times_S X \rightarrow X$. Similarly we get $p_1^n, p_0^n: \Delta_{X/S, (m)}^n \rightarrow X$. As in [Mon02, 2.2.1], we check that p_1 and p_0 are strict morphisms.

If $a \in M_X$, we denote by $\mu_{(m)}(a)$ the unique section of $\ker(\mathcal{O}_{\Delta_{X/S, (m)}}^* \rightarrow \mathcal{O}_X^*)$ such that we get in $M_{X/S, (m)}$ the equality $p_1^*(a) = p_0^*(a)\mu_{(m)}(a)$. We denote by $\mu_{(m)}: M_X \rightarrow \ker(\mathcal{O}_{\Delta_{X/S, (m)}}^* \rightarrow \mathcal{O}_X^*)$ the obvious map. Similarly we get $\mu_{(m)}^n: M_X \rightarrow \ker(\mathcal{O}_{\Delta_{X/S, (m)}^n}^* \rightarrow \mathcal{O}_X^*)$.

2.1.2 (Local description of $\mathcal{P}_{X/S, (m)}$). Suppose that $X \rightarrow S$ is endowed with a log p -basis $(b_\lambda)_{\lambda=1, \dots, r}$. We are in the situation of the proposition 1.2.16 where $u = \Delta$ and f is the left projection $p_0: X \times_S X \rightarrow X$. Indeed, we first remark that $(p_1^*(b_\lambda))_{\lambda=1, \dots, r}$ is a log p -basis of p_0 (indeed, the log p -étaleness property is stable under base change). Since the m -PD envelope compatible with γ of the identity of X is X , proposition 1.2.16 yields the following \mathcal{O}_X - m -PD isomorphism

$$\begin{aligned} \mathcal{O}_X < T_1, \dots, T_r >_{(m)} &\xrightarrow{\sim} \mathcal{P}_{X/S, (m)} \\ T_\lambda &\mapsto \eta_\lambda \end{aligned} \quad (2.1.2.1)$$

where $\eta_\lambda := \eta_{\lambda(m)} := \mu_{(m)}(b_\lambda) - 1$.

2.1.3. Let $g: S' \rightarrow S$ be a morphism of fine log schemes over $\mathbb{Z}/p^{i+1}\mathbb{Z}$, let $(I_{S'}, J_{S'}, \gamma')$ be a quasi-coherent m -PD-ideal of $\mathcal{O}_{S'}$ such that g becomes an m -PD-morphism. Put $X' := X \times_S S'$. We suppose that γ' extends to X' . Then, the m -PD-morphism $\Delta_{X'/S', (m)} \rightarrow \Delta_{X/S, (m)}$ induces the isomorphism $\Delta_{X'/S', (m)} \xrightarrow{\sim} \Delta_{X/S, (m)} \times_S S'$. Indeed, since the morphisms $p_0: \Delta_{X/S, (m)} \rightarrow X$ and $p_0': \Delta_{X'/S', (m)} \rightarrow X'$ are strict, then the morphism $\Delta_{X'/S', (m)} \rightarrow \Delta_{X/S, (m)} \times_S S'$ is strict. Hence, this is sufficient to check that the morphism $g^*\mathcal{P}_{X/S, (m)} \rightarrow \mathcal{P}_{X'/S', (m)}$ is an isomorphism. This can be checked by using the local description of 2.1.2.1.

2.1.4. Using the universal property, for any $m' \geq m$, we get a morphism $\psi_{m, m'}^n: \Delta_{X/S, (m)}^n \rightarrow \Delta_{X/S, (m')}^n$ and then the homomorphism $\psi_{m, m'}^{n*}: \mathcal{P}_{X/S, (m')}^n \rightarrow \mathcal{P}_{X/S, (m)}^n$.

Now, suppose that $X \rightarrow S$ is endowed with a log p -basis $(b_\lambda)_{\lambda=1, \dots, r}$. With the notation of 2.1.2, we have $\psi_{m, m'}^{n*}(\underline{\eta}^{\{k\}}(m')) = \frac{q!}{q'} \underline{\eta}^{\{k\}}(m)$, where $k_\lambda = p^m q_\lambda + r_\lambda$ and $k'_\lambda = p^{m'} q'_\lambda + r'_\lambda$ is the Euclidian division of k_λ by respectively p^m and $p^{m'}$, $\underline{\eta}^{\{k\}}(m) := \prod_{\lambda=1}^r \eta^{\{k_\lambda\}}(m)$, $\underline{q} := \prod_{\lambda=1}^r q_\lambda$ and similarly with some primes.

Notation 2.1.5. We denote by $\mathcal{I}_{X/S,(m)}$ the ideal of the closed immersion $\Delta_{X/S,(m)}$ and $\Omega_{X/S}^1 := \mathcal{I}_{X/S,(m)} / \mathcal{I}_{X/S,(m)}^{\{2\}}$. Thanks to the local description 2.1.2.1 and the local computation of 2.1.4, we notice that this is independent on the level m (which justifies the notation $\Omega_{X/S}^1$). Moreover, for $m \geq 1$ or $p \neq 2$, we have $\mathcal{I}_{X/S,(m)}^{\{2\}} = \mathcal{I}_{X/S,(m)}^2$.

Remark 2.1.6. If X/S has a log p -basis $(b_\lambda)_{\lambda=1,\dots,r}$ then 2.1.2.1 implies that $\Omega_{X/S}^1$ is free of rank r , a basis being given by the images $d\log b_\lambda$ of the η_λ 's. This implies in particular that all p -bases have the same cardinality. We put $\omega_{X/S} := \wedge^r \Omega_{X/S}^1$. More generally, if X/S is log p -smooth, we define $\omega_{X/S}$ in the same way.

Notation 2.1.7. Let \mathcal{E} be an \mathcal{O}_X -module. By convention, $\mathcal{P}_{X/S,(m)}^n \otimes_{\mathcal{O}_X} \mathcal{E}$ means $p_{1*}(\mathcal{P}_{X/S,(m)}^n) \otimes_{\mathcal{O}_X} \mathcal{E}$ and $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S,(m)}^n$ means $\mathcal{E} \otimes_{\mathcal{O}_X} p_{0*}(\mathcal{P}_{X/S,(m)}^n)$. For instance, $\mathcal{P}_{X/S,(m)}^n \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S,(m)}^{n'}$ is $p_{1*}(\mathcal{P}_{X/S,(m)}^n) \otimes_{\mathcal{O}_X} p_{0*}(\mathcal{P}_{X/S,(m)}^{n'})$.

2.1.8. We simply denote by $\Delta_{X/S,(m)}^n \times_X \Delta_{X/S,(m)}^{n'}$ the base change of $p_0^{n'} : \Delta_{X/S,(m)}^{n'} \rightarrow X$ by $p_1^n : \Delta_{X/S,(m)}^n \rightarrow X$. Since p_1^n and $p_0^{n'}$ are flat, the immersion $X \hookrightarrow \Delta_{X/S,(m)}^n \times_X \Delta_{X/S,(m)}^{n'}$ induced by $X \hookrightarrow \Delta_{X/S,(m)}^n$ and $X \hookrightarrow \Delta_{X/S,(m)}^{n'}$ is an object of $\mathcal{C}_{n+n'}^{(m)}$. Hence, from the universal property, we get a unique morphism $\Delta_{X/S,(m)}^n \times_X \Delta_{X/S,(m)}^{n'} \rightarrow \Delta_{X/S,(m)}^{n+n'}$ inducing the commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\quad} & \Delta_{X/S,(m)}^n \times_X \Delta_{X/S,(m)}^{n'} & \xrightarrow{\quad} & X \times_S X \times_S X \\ \parallel & & \downarrow & & \downarrow p_{02} \\ X & \xrightarrow{\quad} & \Delta_{X/S,(m)}^{n+n'} & \xrightarrow{\quad} & X \times_S X. \end{array} \quad (2.1.8.1)$$

We denote by $\delta_{(m)}^{n,n'} : \mathcal{P}_{X/S,(m)}^{n+n'} \rightarrow \mathcal{P}_{X/S,(m)}^n \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S,(m)}^{n'}$ the corresponding morphism.

By replacing p_{02} by p_{01} (resp. p_{12}), we get a unique morphism $\Delta_{X/S,(m)}^n \times_X \Delta_{X/S,(m)}^{n'} \rightarrow \Delta_{X/S,(m)}^{n+n'}$ making commutative the diagram 2.1.8.1. We denote by $q_{0(m)}^{n,n'} : \mathcal{P}_{X/S,(m)}^{n+n'} \rightarrow \mathcal{P}_{X/S,(m)}^n \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S,(m)}^{n'}$ (resp. $q_{1(m)}^{n,n'} : \mathcal{P}_{X/S,(m)}^{n+n'} \rightarrow \mathcal{P}_{X/S,(m)}^n \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S,(m)}^{n'}$) the corresponding morphism (or simply $q_0^{n,n'}$ or q_0). We notice that $q_0^{n,n'} = \pi_{X/S,(m)}^{n+n',n} \otimes 1$ and $q_{1(m)}^{n,n'} = 1 \otimes \pi_{X/S,(m)}^{n+n',n'}$, where $\pi_{X/S,(m)}^{n_1,n_2}$ is the projection $\mathcal{P}_{X/S,(m)}^{n_1} \rightarrow \mathcal{P}_{X/S,(m)}^{n_2}$ for any integers $n_1 \geq n_2$.

2.1.9. We keep the notations of 2.1.8. One checks the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{P}_{X/S,(m)}^{n+n'+n''} & \xrightarrow{\delta_{(m)}^{n,n'+n''}} & \mathcal{P}_{X/S,(m)}^n \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S,(m)}^{n'+n''} \\ \downarrow \delta_{(m)}^{n+n',n''} & & \downarrow \text{Id} \otimes \delta_{(m)}^{n',n''} \\ \mathcal{P}_{X/S,(m)}^{n+n'} \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S,(m)}^{n''} & \xrightarrow{\delta_{(m)}^{n,n'} \otimes \text{Id}} & \mathcal{P}_{X/S,(m)}^n \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S,(m)}^{n'} \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S,(m)}^{n''}. \end{array} \quad (2.1.9.1)$$

2.2 Sheaf of differential operators of level m

Definition 2.2.1. The sheaf of differential operators of level m and order $\leq n$ of f is defined by putting $\mathcal{D}_{X/S,n}^{(m)} := \mathcal{H}om_{\mathcal{O}_X}(p_{0,(m)*} \mathcal{P}_{X/S,(m)}^n, \mathcal{O}_X)$. The sheaf of differential operators of level m of f is defined by putting $\mathcal{D}_{X/S}^{(m)} := \bigcup_{n \in \mathbb{N}} \mathcal{D}_{X/S,n}^{(m)}$.

Let $P \in \mathcal{D}_{X/S,n}^{(m)}$, $P' \in \mathcal{D}_{X/S,n'}^{(m)}$. We define the product $PP' \in \mathcal{D}_{X/S,n+n'}^{(m)}$ to be the composition

$$\mathcal{P}_{X/S,(m)}^{n+n'} \xrightarrow{\delta_{(m)}^{n,n'}} \mathcal{P}_{X/S,(m)}^n \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S,(m)}^{n'} \xrightarrow{\text{Id} \otimes P'} \mathcal{P}_{X/S,(m)}^n \xrightarrow{P} \mathcal{O}_X.$$

Using the commutativity of the square 2.1.9.1, one checks that this product is associative and then that $\mathcal{D}_{X/S}^{(m)}$ is a sheaf of ring.

2.2.2 (Description in local coordinates). Suppose that $X \rightarrow S$ is endowed with a log p -basis $(b_\lambda)_{\lambda=1,\dots,r}$. With the notion of 2.1.4, the elements $\{\eta_{\underline{k}}^{\{k\}(m)}\}_{|\underline{k}| \leq n}$ form a basis of $\mathcal{P}_{X/S,(m)}^n$. The corresponding dual basis of $\mathcal{D}_{X/S,n}^{(m)}$ will be

denoted by $\{\underline{\partial}^{<\underline{k}>(m)}\}_{|\underline{k}|\leq n}$. Let $\epsilon_1, \dots, \epsilon_r$ be the canonical basis of \mathbb{N}^r , i.e. the coordinates of ϵ_λ are 0 except for the i th term which is 1. We put $\partial_\lambda := \underline{\partial}^{<\epsilon_\lambda>(m)}$. We have the same formulas than in [Mon02, 2.3.3]. For instance, for any section $a \in \mathcal{O}_X$, for any $\underline{k}, \underline{k}', \underline{k}'' \in \mathbb{N}^n$,

$$\underline{\partial}^{<\underline{k}>(m)} a = \sum_{\underline{i} \leq \underline{k}} \left\{ \frac{\underline{k}}{\underline{i}} \right\} \underline{\partial}^{<\underline{k}-\underline{i}>(m)}(a) \underline{\partial}^{<\underline{i}>(m)}; \quad (2.2.2.1)$$

$$\underline{\partial}^{<\underline{k}'>(m)} \underline{\partial}^{<\underline{k}''>(m)} = \sum_{\underline{k}=\max\{\underline{k}', \underline{k}''\}}^{\underline{k}'+\underline{k}''} \frac{\underline{k}!}{(\underline{k}' + \underline{k}'' - \underline{k})! (\underline{k} - \underline{k}')! (\underline{k} - \underline{k}'')!} \frac{q_{\underline{k}'}! q_{\underline{k}''}!}{q_{\underline{k}}!} \underline{\partial}^{<\underline{k}>(m)}, \quad (2.2.2.2)$$

where $q_{\underline{k}}$ means the quotient of the Euclidian division of \underline{k} by p^m and similarly with some primes. The left (or right) \mathcal{O}_X -algebra $\mathcal{D}_{X/S}^{(m)}$ is generated by the operators $\partial_\lambda^{<p^j>(m)}$ with $1 \leq \lambda \leq r$, $0 \leq j \leq m$. These formulas yield that $\text{gr}\mathcal{D}_{X/S}^{(m)}$ is commutative and that, when \underline{X} is affine and noetherian, the ring $\Gamma(X, \mathcal{D}_{X/S}^{(m)})$ is left and right noetherian.

2.2.3 (Comparison of the local description of differential operators with or without logarithmic structure). Suppose given log p -étale coordinates $(t_\lambda)_{\lambda=1, \dots, r}$ of X/S (see definition 1.1.28).

1. By 1.2.18, we get the following isomorphism of m -PD- \mathcal{O}_X -algebras

$$\begin{aligned} \mathcal{O}_X < T_1, \dots, T_r >_{(m)} &\xrightarrow{\sim} \mathcal{P}_{X/S, (m)} \\ T_\lambda &\mapsto \tau_\lambda, \end{aligned} \quad (2.2.3.1)$$

where $\tau_\lambda := p_1^*(t_\lambda) - p_0^*(t_\lambda)$. The elements $\{\underline{\tau}^{\{\underline{k}\}}_{(m)}\}_{|\underline{k}|\leq n}$ form a basis of $\mathcal{P}_{X/S, (m)}^n$. The corresponding dual basis of $\mathcal{D}_{X/S, n}^{(m)}$ will be denoted by $\{\underline{\partial}_b^{<\underline{k}>(m)}\}_{|\underline{k}|\leq n}$.

2. Suppose now that the t_λ 's lie in $\Gamma(X, \mathcal{O}_X^*)$. Then, from the remark 1.1.30.1 they are also a finite log p -basis. We have

$$\underline{\tau}^{\{\underline{k}\}}_{(m)} = p_0^*(\underline{t}^{\underline{k}}) \underline{\eta}^{\{\underline{k}\}}_{(m)} \quad \text{and} \quad \underline{\partial}^{<\underline{k}>(m)} = \underline{t}^{\underline{k}} \underline{\partial}_b^{<\underline{k}>(m)}, \quad (2.2.3.2)$$

where $\underline{\eta}$ (resp. $\underline{\partial}^{<\underline{k}>(m)}$) is defined in 2.1.2 (resp. 2.2.2).

3. Suppose now that the t_λ 's lie in $\Gamma(X, \mathcal{O}_X^*)$ and that $X \rightarrow S$ is strict. Since $X \rightarrow S$ is strict, then $\mathcal{P}_{\underline{X}/\underline{S}, (m)}^n = \mathcal{P}_{X/S, (m)}^n$. This yields $\mathcal{D}_{\underline{X}/\underline{S}}^{(m)} = \mathcal{D}_{X/S}^{(m)}$, where $\mathcal{D}_{\underline{X}/\underline{S}}^{(m)}$ is the sheaf of differential operators defined by Berthelot in [Ber96]. Then the local description of 2.2.3.1 extends that given in [Ber96] when $\underline{X}/\underline{S}$ has étale coordinates.

4. Consider the following diagram

$$\begin{array}{ccc} Y & \xrightarrow{b} & A_{\mathbb{N}^r} \times T \\ \downarrow f & & \downarrow \\ X & \xrightarrow{t} & \mathbb{A}^r \times S \end{array} \quad (2.2.3.3)$$

where the right arrow is induced by a morphism of fine log schemes of the form $T \rightarrow S$, the bottom arrow is induced by some log p -étale coordinates $(t_\lambda)_{\lambda=1, \dots, r}$ and where the top arrow is induced by a log p -étale basis $(b_\lambda)_{\lambda=1, \dots, r}$. Let $\underline{\eta}$ (resp. $\underline{\partial}^{<\underline{k}>(m)}$) be the element constructed from $(b_\lambda)_{\lambda=1, \dots, r}$ as defined in 2.1.2 (resp. 2.2.2). Then functoriality morphisms $f^* \mathcal{P}_{X/S, (m)}^n \rightarrow \mathcal{P}_{Y/T, (m)}^n$ and $\mathcal{D}_{Y/T}^{(m)} \rightarrow f^* \mathcal{D}_{X/S}^{(m)}$ (see 2.2.5) are explicitly described by

$$\underline{\tau}^{\{\underline{k}\}}_{(m)} \mapsto \underline{t}^{\underline{k}} \underline{\eta}^{\{\underline{k}\}}_{(m)} \quad \text{and} \quad \underline{\partial}^{<\underline{k}>(m)} \mapsto \underline{t}^{\underline{k}} \underline{\partial}_b^{<\underline{k}>(m)}. \quad (2.2.3.4)$$

2.2.4. For any $m' \geq m$, from the homomorphisms $\psi_{m, m'}^{n*}: \mathcal{P}_{X/S, (m')}^n \rightarrow \mathcal{P}_{X/S, (m)}^n$ of 2.1.4, we get by duality, the homomorphisms $\rho_{m', m}: \mathcal{D}_{X/S}^{(m)} \rightarrow \mathcal{D}_{X/S}^{(m')}$. Let $k_\lambda = p^m q_\lambda + r_\lambda$ and $k'_\lambda = p^{m'} q'_\lambda + r'_\lambda$ be the Euclidian division of k by respectively p^m and $p^{m'}$. Now, suppose that $X \rightarrow S$ is endowed with a log p -basis $(b_\lambda)_{\lambda=1, \dots, r}$. With its notation, we get from 2.1.4 the equality $\rho_{m', m}(\underline{\partial}^{<\underline{k}>(m)}) = \frac{q!}{q'}! \underline{\partial}^{<\underline{k}>(m')}$.

2.2.5. Let S' be a fine log scheme over $\mathbb{Z}/p^{i+1}\mathbb{Z}$ and let $(I_{S'}, J_{S'}, \gamma')$ be a quasi-coherent m -PD-ideal of $\mathcal{O}_{S'}$. Let $g: S' \rightarrow S$ be an m -PD-morphism (with respect to these m -PD-structures). Consider the commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow \pi_{X'} & & \downarrow \pi_X \\ S' & \xrightarrow{g} & S \end{array} \quad (2.2.5.1)$$

such that π_X and $\pi_{X'}$ are log p -smooth, γ (resp. γ') extends to X (resp. X'). Using the universal property of the m -PD envelope, we get the m -PD-morphism $f^*\mathcal{P}_{X/S,(m)}^n \rightarrow \mathcal{P}_{X'/S',(m)}^n$. This yields the morphism $\mathcal{D}_{X'/S',n}^{(m)} \rightarrow f^*\mathcal{D}_{X/S,n}^{(m)}$ and then $\mathcal{D}_{X'/S'}^{(m)} \rightarrow f^*\mathcal{D}_{X/S}^{(m)}$.

When the diagram 2.2.5.1 is cartesian (in the category of fine log schemes), the morphism $f^*\mathcal{P}_{X/S,(m)}^n \rightarrow \mathcal{P}_{X'/S',(m)}^n$ is in fact an isomorphism and then so is $\mathcal{D}_{X'/S'}^{(m)} \rightarrow f^*\mathcal{D}_{X/S}^{(m)}$.

When $(S', I_{S'}, J_{S'}, \gamma') = (S, I_S, J_S, \gamma)$, and f is log p -étale, then the morphism $f^*\mathcal{P}_{X/S,(m)}^n \rightarrow \mathcal{P}_{X'/S,(m)}^n$ is in fact an isomorphism and then so is $\mathcal{D}_{X'/S}^{(m)} \rightarrow f^*\mathcal{D}_{X/S}^{(m)}$.

2.3 Logarithmic PD stratification of level m

One can follow Berthelot's construction of PD stratifications of level m and check properties analogous to those of [Ber02] or [Ber96]. Let us give a quick and partial exposition. Even if one might consider the étale topology, an \mathcal{O}_X -module will mean an \mathcal{O}_X -module for the Zariski topology.

Definition 2.3.1. Let \mathcal{E} be an \mathcal{O}_X -module. An m -PD-stratification (or a PD-stratification of level m) is the data of a family of compatible (with respect to the projections $\pi_{X/S,(m)}^{n+1,n}$) $\mathcal{P}_{X/S,(m)}^n$ -linear isomorphisms

$$\epsilon_n^\mathcal{E}: \mathcal{P}_{X/S,(m)}^n \otimes_{\mathcal{O}_X} \mathcal{E} \xrightarrow{\sim} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S,(m)}^n$$

satisfying the following conditions:

1. $\epsilon_0^\mathcal{E} = \text{Id}_\mathcal{E}$;
2. for any n, n' , the diagram

$$\begin{array}{ccc} \mathcal{P}_{X/S,(m)}^n \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S,(m)}^{n'} \otimes_{\mathcal{O}_X} \mathcal{E} & \xrightarrow[\sim]{\delta_{(m)}^{n,n'} * (\epsilon_{n+n'}^\mathcal{E})} & \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S,(m)}^n \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S,(m)}^{n'} \\ & \searrow \scriptstyle q_{1(m)}^{n,n'} * (\epsilon_{n+n'}^\mathcal{E}) \quad \nearrow \scriptstyle q_{0(m)}^{n,n'} * (\epsilon_{n+n'}^\mathcal{E}) & \\ & \mathcal{P}_{X/S,(m)}^n \otimes_{\mathcal{O}_X} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S,(m)}^{n'} & \end{array}$$

is commutative

Proposition 2.3.2. Let \mathcal{E} be an \mathcal{O}_X -module. The following datas are equivalent :

1. A structure of left $\mathcal{D}_{X/S}^{(m)}$ -module on \mathcal{E} extending its structure of \mathcal{O}_X -module.
2. A family of compatible \mathcal{O}_X -linear homomorphisms $\theta_n^\mathcal{E}: \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S,(m)}^n$ such that $\theta_0^\mathcal{E} = \text{Id}_\mathcal{E}$ and for any integers n, n' the diagram

$$\begin{array}{ccc} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S,(m)}^n & \xrightarrow{\text{Id} \otimes \delta_{(m)}^{n,n'}} & \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S,(m)}^n \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S,(m)}^{n'} \\ \theta_{n+n'}^\mathcal{E} \uparrow & & \theta_n^\mathcal{E} \otimes \text{Id} \uparrow \\ \mathcal{E} & \xrightarrow{\theta_{n'}^\mathcal{E}} & \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S,(m)}^{n'} \end{array} \quad (2.3.2.1)$$

is commutative.

3. An m -PD-stratification on \mathcal{E} .

An \mathcal{O}_X -linear morphism $\phi: \mathcal{E} \rightarrow \mathcal{F}$ between two left $\mathcal{D}_{X/S}^{(m)}$ -modules is $\mathcal{D}_{X/S}^{(m)}$ -linear if and only if it commutes with the homomorphisms θ_n (resp. ϵ_n).

Proof. The proof is identical to that of [Mon02, 2.6.1] or [Ber96, 2.3.2]. \square

2.3.3. If $X \rightarrow S$ is endowed with a log p -basis $(b_\lambda)_{\lambda=1,\dots,n}$ then for any $x \in \mathcal{E}$ we have the Taylor development

$$\theta_n^\mathcal{E}(x) = \sum_{|\underline{k}| \leq n} \underline{\partial}^{<\underline{k}>(m)} \cdot x \otimes \underline{\eta}^{\{\underline{k}\}}. \quad (2.3.3.1)$$

In order to define overconvergent isocrystals in our context (see 3.2.5), we will need the following definition and proposition.

Definition 2.3.4. Let \mathcal{B} be a commutative \mathcal{O}_X -algebra endowed with a structure of left $\mathcal{D}_{X/S}^{(m)}$ -module. We say that the structure of left $\mathcal{D}_{X/S}^{(m)}$ -module on \mathcal{B} is compatible with its structure of \mathcal{O}_X -algebra if the isomorphisms $\epsilon_n^\mathcal{B}$ are isomorphisms of $\mathcal{P}_{X/S,(m)}^n$ -algebras. This compatibility is equivalent to the following condition : for any $f, g \in \mathcal{B}$ and $\underline{k} \in \mathbb{N}^d$,

$$\underline{\partial}^{<\underline{k}>(m)}(fg) = \sum_{\underline{i} \leq \underline{k}} \left\{ \begin{matrix} \underline{k} \\ \underline{i} \end{matrix} \right\} \underline{\partial}^{<\underline{i}>(m)}(f) \underline{\partial}^{<\underline{k}-\underline{i}>(m)}(g).$$

Proposition 2.3.5. Let \mathcal{B} be a commutative \mathcal{O}_X -algebra endowed with a compatible structure of left $\mathcal{D}_{X/S}^{(m)}$ -module. Then there exists on the tensor product $\mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S}^{(m)}$ a unique structure of rings satisfying the following conditions

1. the canonical morphisms $\mathcal{B} \rightarrow \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S}^{(m)}$ and $\mathcal{D}_{X/S}^{(m)} \rightarrow \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S}^{(m)}$ are homomorphisms of sheaf of rings,
2. if $X \rightarrow S$ is endowed with a log p -basis $(b_\lambda)_{\lambda=1,\dots,n}$, then, for any $b \in \mathcal{B}$ and $\underline{k} \in \mathbb{N}^n$, we have $(b \otimes 1)(1 \otimes P) = b \otimes P$ and

$$(1 \otimes \underline{\partial}^{<\underline{k}>(m)})(b \otimes 1) = \sum_{\underline{i} \leq \underline{k}} \left\{ \begin{matrix} \underline{k} \\ \underline{i} \end{matrix} \right\} \underline{\partial}^{<\underline{i}>(m)}(b) \otimes \underline{\partial}^{<\underline{k}-\underline{i}>(m)}.$$

If $\mathcal{B} \rightarrow \mathcal{B}'$ is a morphism of \mathcal{O}_X -algebras with compatible structure of left $\mathcal{D}_{X/S}^{(m)}$ -modules, then the induced morphism $\mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S}^{(m)} \rightarrow \mathcal{B}' \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S}^{(m)}$ is a homomorphism of rings.

Proof. We copy [Ber96, 2.3.5]. \square

2.4 Logarithmic transposition

We suppose that $X \rightarrow S$ is endowed with a log p -basis $(b_\lambda)_{\lambda=1,\dots,n}$.

Notation 2.4.1. For any $\underline{k} \in \mathbb{N}^n$, we set

$$\tilde{\underline{\partial}}^{<\underline{k}>(m)} := (-1)^{|\underline{k}|} \sum_{1 \leq i \leq \underline{k}} \underline{\partial}^{<\underline{i}>(m)} \left\{ \begin{matrix} \underline{k} \\ \underline{i} \end{matrix} \right\} q_{\underline{k}-\underline{i}}! \left(\frac{\underline{k}-1}{\underline{k}-\underline{i}} \right), \quad (2.4.1.1)$$

where q_k means the quotient of the Euclidian division of k by p^m . For any differential operator P of the form $P = \sum_{\underline{k}} a_{\underline{k}} \underline{\partial}^{<\underline{k}>(m)}$, we set $\tilde{P} := \sum_{\underline{k}} \tilde{\underline{\partial}}^{<\underline{k}>(m)} a_{\underline{k}}$. We say that \tilde{P} is the logarithmic transposition of P .

Remark 2.4.2 (Comparison between transposition with or without logarithmic structure). We suppose that f is p -smooth and that $b_1, \dots, b_n \in \mathcal{O}_X^*$ (i.e. they form a p -basis). In that case, we prefer to denote $t_\lambda := b_\lambda$. With the notation of 2.2.3, any differential operator P of $\mathcal{D}_{X/S}^{(m)}$ can be written of the form $P = \sum_{\underline{k}} a_{\underline{k}} \underline{\partial}_b^{<\underline{k}>(m)}$. We can extend the non logarithmic transposition as defined by Berthelot (see [Ber00, 1.3]) to our context by putting

$${}^t P := \sum_{\underline{k}} (-1)^{|\underline{k}|} \underline{\partial}_b^{<\underline{k}>(m)} a_{\underline{k}}.$$

Then we have

$$\tilde{P} = \underline{t} {}^t P \frac{1}{\underline{t}}.$$

Indeed, it is enough to check it when $P = \underline{\partial}^{<\underline{k}>(m)}$. But, the definition of 2.4.1.1 was precisely introduced to get $\tilde{\underline{\partial}}^{<\underline{k}>(m)} = \underline{t} {}^t \underline{\partial}^{<\underline{k}>(m)} \frac{1}{\underline{t}}$.

One reason to introduce the logarithmic transposition is the formula 3.3.2.1.

Proposition 2.4.3. *For any differential operators P and Q , we have $\widetilde{PQ} = \widetilde{Q}\tilde{P}$.*

Proof. 0) When $P \in \mathcal{O}_X$, the proposition is obvious.

1) Suppose that $P = \underline{\partial}^{<\underline{k}>(m)}$ and $Q = a \in \mathcal{O}_X$. For any tuples $\underline{i}, \underline{j} \in \mathbb{N}^n$ so that $\underline{i} \leq \underline{j}$, we put $\alpha_{\underline{i}, \underline{j}} := (-1)^{|\underline{j}|} \left\{ \frac{\underline{j}}{\underline{i}} \right\} q_{\underline{j}-\underline{i}}! \left(\frac{\underline{j}-1}{\underline{j}-\underline{i}} \right)$ if $\underline{1} \leq \underline{i}$ and $\alpha_{\underline{i}, \underline{j}} := 0$ otherwise. For any $\underline{h}, \underline{i}, \underline{j}, \underline{k} \in \mathbb{N}^n$ so that $\underline{h} \leq \underline{i} \leq \underline{j} \leq \underline{k}$, we put $P_{\underline{h}, \underline{i}, \underline{j}, \underline{k}} := \alpha_{\underline{i}, \underline{j}} \left\langle \frac{\underline{i}}{\underline{h}} \right\rangle \left\langle \frac{\underline{k}}{\underline{j}} \right\rangle \underline{\partial}^{<\underline{i}-\underline{h}>(m)} \underline{\partial}^{<\underline{k}-\underline{j}>(m)}$. On one side we have $a \tilde{\underline{\partial}}^{<\underline{k}>(m)} = \sum_{\underline{h} \leq \underline{k}} \alpha_{\underline{h}, \underline{k}} a \underline{\partial}^{<\underline{h}>(m)}$ and on the other side using twice the formula 2.2.2.1 we compute $(\underline{\partial}^{<\underline{k}>(m)} a)^\sim = \sum_{\underline{h} \leq \underline{i} \leq \underline{j} \leq \underline{k}} P_{\underline{h}, \underline{i}, \underline{j}, \underline{k}}(a) \underline{\partial}^{<\underline{h}>(m)}$. Hence, when \underline{h} and \underline{k} are fixed, this is sufficient to check $\alpha_{\underline{h}, \underline{k}} = \sum_{\underline{h} \leq \underline{i} \leq \underline{j} \leq \underline{k}} P_{\underline{h}, \underline{i}, \underline{j}, \underline{k}}$. Since the coefficients of the differential operators $P_{\underline{h}, \underline{i}, \underline{j}, \underline{k}}$ are integers, shrinking X if necessary, we reduce to check the equality when the log structures are trivial. With the remark 2.4.2, we obtained the desired equality.

2) When $P = \underline{\partial}^{<\underline{k}>(m)}$ and $Q = \underline{\partial}^{<\underline{k}'>(m)}$, using the formula 2.2.2.1, we check the equality $\widetilde{PQ} = \tilde{\underline{\partial}}^{<\underline{k}'>(m)} \tilde{\underline{\partial}}^{<\underline{k}>(m)}$. Indeed, with the remark 2.4.2, we notice that the formula we have to check is the same as that obtained using the analogy $\tilde{P} = \underline{t} {}^t P \frac{1}{\underline{t}}$.

3) Suppose $P = \underline{\partial}^{<\underline{k}'>(m)}$ and $Q = \underline{\partial}^{<\underline{k}''>(m)} a$, with $a \in \mathcal{O}_X$. From 2.2.2.1, we have the equality $\underline{\partial}^{<\underline{k}'>(m)} \underline{\partial}^{<\underline{k}''>(m)} a = \sum_{\underline{k}=\max\{\underline{k}', \underline{k}''\}}^{\underline{k}'+\underline{k}''} \beta_{\underline{k}, \underline{k}', \underline{k}''} \underline{\partial}^{<\underline{k}>(m)} a$, where $\beta_{\underline{k}, \underline{k}', \underline{k}''} := \frac{k!}{(\underline{k}'+\underline{k}''-\underline{k})!(\underline{k}-\underline{k}')!(\underline{k}-\underline{k}'')!} \frac{q_{\underline{k}'}! q_{\underline{k}''}!}{q_{\underline{k}}!} \in \mathbb{Z}$. Hence, from the step 1), we get $(\underline{\partial}^{<\underline{k}'>(m)} \underline{\partial}^{<\underline{k}''>(m)} a)^\sim = a \sum_{\underline{k}=\max\{\underline{k}', \underline{k}''\}}^{\underline{k}'+\underline{k}''} \beta_{\underline{k}, \underline{k}', \underline{k}''} \tilde{\underline{\partial}}^{<\underline{k}>(m)}$. Applying \sim to the equality 2.2.2.1 and using the step 1), we obtain $\sum_{\underline{k}=\max\{\underline{k}', \underline{k}''\}}^{\underline{k}'+\underline{k}''} \beta_{\underline{k}, \underline{k}', \underline{k}''} \tilde{\underline{\partial}}^{<\underline{k}>(m)} = \tilde{\underline{\partial}}^{<\underline{k}''>(m)} \tilde{\underline{\partial}}^{<\underline{k}'>(m)}$. Again using the step 1), this yields $(\underline{\partial}^{<\underline{k}'>(m)} \underline{\partial}^{<\underline{k}''>(m)} a)^\sim = (\tilde{\underline{\partial}}^{<\underline{k}''>(m)} a)^\sim \tilde{\underline{\partial}}^{<\underline{k}'>(m)}$.

4) By additivity, using part 0) and part 4), we check the proposition. \square

Remark 2.4.4 (Logarithmic transposition at the level 0). At the level 0, any differential operator of $\mathcal{D}_{X/S}^{(0)}$ can be written uniquely in the form $P = \sum_{\underline{k}} a_{\underline{k}} \underline{\partial}^{\underline{k}}$, where $\underline{\partial}^{\underline{k}}$ has not to be confounded with $\underline{\partial}^{<\underline{k}>(0)}$. Since $\tilde{\underline{\partial}} = -\underline{\partial}$, we get $\tilde{P} = \sum_{\underline{k}} (-1)^{|\underline{k}|} \underline{\partial}^{\underline{k}} a_{\underline{k}}$.

Proposition 2.4.5. *For any differential operator P , we have $\tilde{\tilde{P}} = P$.*

Proof. Using 2.4.3, we reduce to the case where $P = \underline{\partial}^{<\underline{k}>(m)}$, which is left to the reader. \square

Remark 2.4.6. Recall that from 2.2.2.1, we have $\underline{\partial}^{<\underline{k}>(m)} a = \sum_{\underline{i} \leq \underline{k}} \left\{ \frac{\underline{k}}{\underline{i}} \right\} \underline{\partial}^{<\underline{k}-\underline{i}>(m)}(a) \underline{\partial}^{<\underline{i}>(m)}$. Using 2.4.3, this yields

$$a \tilde{\underline{\partial}}^{<\underline{k}>(m)} = \sum_{\underline{i} \leq \underline{k}} \tilde{\underline{\partial}}^{<\underline{i}>(m)} \left\{ \frac{\underline{k}}{\underline{i}} \right\} \underline{\partial}^{<\underline{k}-\underline{i}>(m)}(a). \quad (2.4.6.1)$$

In the formula 2.4.6.1, beware that we can not replace $\tilde{\underline{\partial}}$ by $\underline{\partial}$.

2.4.7. The logarithmic transposition commutes with the canonical morphism $\mathcal{D}_{X/S}^{(m)} \rightarrow \mathcal{D}_{X/S}^{(m+1)}$.

3 Differential operators over log p -smooth fine log formal schemes

The principal ideal (π) of \mathcal{V} is endowed with a canonical m -PD-structure, which we will denote by γ_π . We recall that Shiho introduced the notion of log formal \mathcal{V} -schemes (see [Shi00, 2.1.1.(4)]) as follows: A log formal \mathcal{V} -scheme \mathfrak{X} is a formal \mathcal{V} -scheme $\underline{\mathfrak{X}}$ endowed with a logarithmic structure $\alpha: M_{\underline{\mathfrak{X}}} \rightarrow \mathcal{O}_{\underline{\mathfrak{X}}}$, where $\mathcal{O}_{\underline{\mathfrak{X}}} := \mathcal{O}_{\underline{\mathfrak{X}}}$. When the logarithmic structure is fine (resp. fine and saturated), we will say that \mathfrak{X} is a fine log formal \mathcal{V} -scheme (resp. a fs log formal \mathcal{V} -scheme). Let \mathcal{S} be a fine log formal \mathcal{V} -scheme such that the underlying formal \mathcal{V} -scheme is $\text{Spf } \mathcal{V}$. A fine log formal \mathcal{S} -scheme \mathfrak{X} will be a morphism of fine log formal \mathcal{V} -schemes of the form $\mathfrak{X} \rightarrow \mathcal{S}$.

If \mathfrak{X} is a fine log formal \mathcal{V} -scheme, then we denote by X_i the fine log $\mathcal{V}/\pi^{n+1}\mathcal{V}$ -scheme so that $\underline{X}_i := \mathfrak{X} \times_{\mathrm{Spf}(\mathcal{V})} \mathrm{Spec}(\mathcal{V}/\pi^{n+1}\mathcal{V})$ and the morphism $X_i \rightarrow \mathfrak{X}$ is strict. For $i = 0$, we can simply denote X_0 by X . If $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ is a morphism of fine log formal \mathcal{V} -schemes, then we denote by $f_i: X_i \rightarrow Y_i$ the induced morphism of fine log-schemes over $\mathcal{V}/\pi^{n+1}\mathcal{V}$. We remark that if $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ is a morphism of fine (resp. fs) log formal \mathcal{S} -schemes, then $f_i: X_i \rightarrow Y_i$ is a morphism of fine (resp. fs) log \mathcal{S}_i -schemes.

3.1 Sheaf of differential operators of level m over log p -smooth fine log formal \mathcal{S} -schemes

3.1.1 (Charts for log formal \mathcal{V} -schemes). Let P be a fine monoid and $\mathcal{V}\{P\}$ be the p -adic completion of $\mathcal{V}[P]$. We denote by \widehat{A}_P the fine log formal \mathcal{V} scheme whose underlying formal \mathcal{V} -scheme is $\mathrm{Spf}(\mathcal{V}\{P\})$ and whose log structure is the log structure associated with the pre-log structure induced canonically by $P \rightarrow \mathcal{V}\{P\}$.

Let \mathfrak{X} be a fine log formal \mathcal{S} -scheme. We denote by $P_{\mathfrak{X}}$ the corresponding sheaf of monoids over \mathfrak{X} . Following Shiho's definition of [Shi00, 2.1.7], a chart of \mathfrak{X} is a morphism of monoids $\alpha: P_{\mathfrak{X}} \rightarrow \mathcal{O}_{\mathfrak{X}}$ whose associated log structure is isomorphic to $M_{\mathfrak{X}} \rightarrow \mathcal{O}_{\mathfrak{X}}$. A chart of \mathfrak{X} is equivalent to the data of a strict morphism of the form $\mathfrak{X} \rightarrow \widehat{A}_P$.

Definition 3.1.2. A “topologically (p) -nilpotent formal log thickening” (resp. topologically nilpotent log thickening) is a morphism $u: \mathfrak{U} \hookrightarrow \mathfrak{T}$ of formal log \mathcal{S} -schemes such that for any i the \mathcal{S}_i -immersion u_i is an object of $\mathcal{T}hick_{(p)}$ (resp. is nilpotent). We denote by $\mathcal{T}hick_{(p)}$ (resp. $\mathfrak{N}ilp$) the category of topologically (p) -nilpotent formal log thickenings (resp. topologically nilpotent formal log thickenings). We denote by $\mathcal{T}hick_{(p)}^{\mathrm{sat}}$ (resp. $\mathfrak{N}ilp^{\mathrm{sat}}$) the full subcategory of $\mathcal{T}hick_{(p)}$ (resp. $\mathfrak{N}ilp$) of objects which are also morphisms of fs formal \mathcal{V} -schemes.

Definition 3.1.3. Let $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism of fine formal log \mathcal{S} -schemes.

1. We say that f is “fine formally log étale” (resp. “fs formally log étale”, resp. “log p -étale”, resp. “fs log p -étale”) if it satisfies the following universal property: for any commutative diagram of fine log formal schemes of the form

$$\begin{array}{ccc} \mathfrak{U} & \xrightarrow{u} & \mathfrak{X} \\ \downarrow \iota & & \downarrow f \\ \mathfrak{T} & \xrightarrow{v} & \mathfrak{Y} \end{array} \quad (3.1.3.1)$$

such that ι is an object of $\mathfrak{N}ilp$ (resp. $\mathfrak{N}ilp^{\mathrm{sat}}$, resp. $\mathcal{T}hick_{(p)}$, resp. $\mathcal{T}hick_{(p)}^{\mathrm{sat}}$), there exists a unique morphism $\tilde{u}: \mathfrak{T} \rightarrow \mathfrak{X}$ such that $\tilde{u} \circ \iota = u$ and $f \circ \tilde{u} = v$.

2. We say that f is log étale (resp. fine log relatively perfect, resp. fs log relatively perfect) if for any integer i the morphism f_i is log étale (resp. fine log relatively perfect, resp. fs log relatively perfect).

Remark 3.1.4. We remark that our definition of log étaleness was named by Shiho formal log étaleness (see [Shi00, 2.2.2]). We hope there will be no confusion.

Lemma 3.1.5. Let \mathfrak{X} be a fine log formal \mathcal{S} -scheme. Then, in the category of fine log formal \mathcal{S} -schemes, \mathfrak{X} is the inductive limit of the system $(X_i)_i$.

Proof. From [Gro60, I.10.6.1], \mathfrak{X} is the inductive limit of the system $(\underline{X}_i)_i$. It remains to check that the canonical map $M_{\mathfrak{X}} \rightarrow \varprojlim_i M_{X_i}$ is an isomorphism. Since this is étale local on \mathfrak{X} and since \mathfrak{X} is fine then we can suppose there exists a fine monoid P and a morphism of sheaves of monoids $\alpha: P_{\mathfrak{X}} \rightarrow \mathcal{O}_{\mathfrak{X}}$ (here $P_{\mathfrak{X}}$ means the sheaf associated to the constant presheaf of P over \mathfrak{X}) which induces the isomorphism of sheaves of monoids $P_{\mathfrak{X}} \oplus_{\alpha^{-1}(\mathcal{O}_{\mathfrak{X}}^*)} \mathcal{O}_{\mathfrak{X}}^* \xrightarrow{\sim} M_{\mathfrak{X}}$. Let $i \geq 0$ be an integer. We get the morphism of sheaves of monoids $\alpha_i: P_{X_i} \rightarrow \mathcal{O}_{X_i}$ which induces the isomorphism $P_{X_i} \oplus_{\alpha_i^{-1}(\mathcal{O}_{X_i}^*)} \mathcal{O}_{X_i}^* \xrightarrow{\sim} M_{X_i}$. Hence, we reduce to prove that the canonical map $P_{\mathfrak{X}} \oplus_{\alpha^{-1}(\mathcal{O}_{\mathfrak{X}}^*)} \mathcal{O}_{\mathfrak{X}}^* \rightarrow \varprojlim_i P_{X_i} \oplus_{\alpha_i^{-1}(\mathcal{O}_{X_i}^*)} \mathcal{O}_{X_i}^*$ is an isomorphism. We put $\mathcal{F}_i := P_{X_i} \oplus_{\alpha_i^{-1}(\mathcal{O}_{X_i}^*)} \mathcal{O}_{X_i}^*$ where “ \oplus ” means that the amalgamated sum is computed in the category of pre-sheaves. We put $\mathcal{E}_i := P_{X_i} \oplus \mathcal{O}_{X_i}^*$, $\theta: \mathcal{E}_i \rightarrow \mathcal{F}_i$ the canonical surjective morphism, $\mathcal{G}_i := P_{X_i} \oplus_{\alpha_i^{-1}(\mathcal{O}_{X_i}^*)} \mathcal{O}_{X_i}^*$ and $\epsilon: \mathcal{F}_i \rightarrow \mathcal{G}_i$ the canonical morphism from a pre-sheaf to its associated sheaf. We put $\phi := \epsilon \circ \theta$. We denote by $\pi: \mathcal{O}_{X_{i+1}}^* \rightarrow \mathcal{O}_{X_i}^*$, $\pi: \mathcal{E}_{i+1} \rightarrow \mathcal{E}_i$, $\pi: \mathcal{F}_{i+1} \rightarrow \mathcal{F}_i$, $\pi: \mathcal{G}_{i+1} \rightarrow \mathcal{G}_i$ the canonical projections. Let $\mathfrak{U} \rightarrow \mathfrak{X}$ be an étale map. As a topological space, we remark that $|\mathfrak{U}| = |U_i|$.

- 1) Let $s_{i+1} \in \mathcal{F}_{i+1}(U_{i+1})$ and $s_i := \pi(s_{i+1}) \in \mathcal{F}_i(U_i)$. Then the canonical map $\pi: \theta^{-1}(s_{i+1}) \rightarrow \theta^{-1}(s_i)$ is a bijection.
 - a) We check the injectivity. Let $(x, a), (x', a') \in \theta^{-1}(s_{i+1})$ such that $\pi(x, a) = \pi(x', a')$. The latter equality yields $x = x'$. Since P is integral, $\theta(x, a) = \theta(x, a')$ implies $a = a'$.

b) We check the surjectivity. Let $(y, b) \in \theta^{-1}(s_i)$. We remark that $\alpha^{-1}(\mathcal{O}_{\mathfrak{X}}^*) = \alpha_i^{-1}(\mathcal{O}_{X_i}^*)$ and we denote it by Q . Since θ is an epimorphism (in the category of presheaves) then there exists $(x, a) \in \theta^{-1}(s_{i+1})$. Since $\pi(x, a) = (x, \pi(a)) \in \theta^{-1}(s_i)$, there exists $q, q' \in Q(U_i)$ such that $\pi(a)\alpha_i(q) = b\alpha_i(q')$ and $xq' = yq$. Set $a' := a\alpha_{i+1}(q)\alpha_{i+1}(q')^{-1}$. Then $\pi(a') = b$ and $\theta(x, a) = \theta(y, a')$, i.e. $\pi(y, a') = (y, b)$ and $(y, a') \in \theta^{-1}(s_{i+1})$.

2) Let $t_{i+1} \in \mathcal{G}_{i+1}(U_{i+1})$ and $t_i := \pi(t_{i+1}) \in \mathcal{G}_i(U_i)$. Then the canonical map $\pi: \phi^{-1}(t_{i+1}) \rightarrow \phi^{-1}(t_i)$ is a bijection.

a) We check the injectivity. Let $r, r' \in \phi^{-1}(t_{i+1})$ such that $\pi(r) = \pi(r')$. There exists an étale covering $(\mathfrak{U}_\lambda \rightarrow \mathfrak{U})_\lambda$ of \mathfrak{U} such that $\theta(r)|_{U_\lambda} = \theta(r')|_{U_\lambda}$. From 1) (applied for \mathfrak{U}_λ instead of \mathfrak{U}), this yields $r|_{U_\lambda} = r'|_{U_\lambda}$. Hence, $r = r'$.

b) We check the surjectivity. Let $r \in \phi^{-1}(t_i)$. Put $s := \theta(r)$. There exist an étale covering $(\mathfrak{U}_\lambda \rightarrow \mathfrak{U})_\lambda$ of \mathfrak{U} and sections $s_\lambda \in \mathcal{F}_{i+1}(U_\lambda)$ such that $\epsilon(s_\lambda) = t_{i+1}|_{U_\lambda}$ and $\pi(s_\lambda) = s|_{U_\lambda}$. From 1.b), there exists $r_\lambda \in \mathcal{E}_{i+1}(U_\lambda)$ such that $\pi(r_\lambda) = r|_{U_\lambda}$ and $\theta(r_\lambda) = s_\lambda$. Hence, $\pi(r_\lambda) = r|_{U_\lambda}$ and $\phi(r_\lambda) = t_{i+1}|_{U_\lambda}$. From 2.a), this yields that $(r_\lambda)_\lambda$ come from a section of $\mathcal{E}_{i+1}(U_{i+1})$.

3) Using 2), this becomes obvious that the canonical map $P_{\mathfrak{X}} \oplus_{\alpha^{-1}(\mathcal{O}_{\mathfrak{X}}^*)} \mathcal{O}_{\mathfrak{X}}^* \rightarrow \varprojlim_i P_{X_i} \oplus_{\alpha_i^{-1}(\mathcal{O}_{X_i}^*)} \mathcal{O}_{X_i}^*$ is an isomorphism. \square

Proposition 3.1.6. *Let $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism of fine formal log \mathcal{S} -schemes.*

1. *Then f is fine formally log étale (resp. fs formally log étale) if and only if, for any integer i , the morphism f_i is fine formally log étale (resp. fs formally log étale).*
2. *The morphism f is log p -étale (resp. fs log p -étale) if and only if f is fine formally log étale (resp. f is fs formally log étale) and f_0 is log p -étale (resp. fs log p -étale).*
3. *The morphism f is fine log relatively perfect (resp. fs log relatively perfect) if and only if f is fine formally log étale (resp. f is fs formally log étale) and f_0 is fine log relatively perfect (resp. fs log relatively perfect).*
4. *If f is log étale then f is log p -étale and fs log relatively perfect. If f is fine (resp. fs) log relatively perfect then f is (resp. fs) log p -étale.*

Proof. With 1.1.25.1, this is a consequence of 3.1.5 or this is abstract nonsense or this is straightforward. \square

Definition 3.1.7. As in 1.1.4 we define the category \mathfrak{C} of \mathcal{S} -immersions of fine formal log \mathcal{S} -schemes. As in 1.2.1, we define the categories $\mathfrak{C}^{(m)}$ (resp. $\mathfrak{C}_n^{(m)}$) whose objects are pairs (u, δ) where u is an exact closed \mathcal{S} -immersion of fine log \mathcal{S} -schemes and δ is an m -PD-structure on the ideal \mathcal{I} defining u which is compatible with γ_π (resp. and such that $\mathcal{I}^{(n+1)}(m) = 0$) and whose morphisms $(u', \delta') \rightarrow (u, \delta)$ are morphisms $u' \rightarrow u$ of \mathfrak{C} which are compatible with the m -PD-structures δ and δ' .

Remark 3.1.8. Let (u, δ) be an object of $\mathfrak{C}^{(m)}$ (resp. $\mathfrak{C}_n^{(m)}$). Then u is an object of $\mathfrak{Thick}_{(p)}$.

Proposition 3.1.9. 1. *The canonical functor $\mathfrak{C}^{(m)} \rightarrow \mathfrak{C}$ (resp. $\mathfrak{C}_n^{(m)} \rightarrow \mathfrak{C}$) has a right adjoint, which we will denote by $P_{(m)}: \mathfrak{C} \rightarrow \mathfrak{C}^{(m)}$ (resp. $P_{(m)}^n: \mathfrak{C} \rightarrow \mathfrak{C}_n^{(m)}$).*

2. *Let u be an object of \mathfrak{C} . The source of $P_{(m)}(u)$ (resp. $P_{(m)}^n(u)$) is the source of u .*

Proof. The first assertion is a consequence of 3.1.5 and 1.2.11. Since γ_π extends to any log formal \mathcal{S} -schemes (because the ideal of the m -PD-structure γ_π is locally principal, see [Ber96, 1.3.2.c]), we get the second assertion. \square

3.1.10. Let u be an object of \mathfrak{C} . We call $P_{(m)}(u)$ the m -PD-envelope compatible with γ_π of u . We sometimes denote abusively by $P_{(m)}(u)$ (resp. $P_{(m)}^n(u)$) the target of the arrow $P_{(m)}(u)$ (resp. $P_{(m)}^n(u)$).

Definition 3.1.11. Let $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism of fine formal log \mathcal{S} -schemes.

1. We say that a finite set $(b_\lambda)_{\lambda=1, \dots, r}$ of elements of $\Gamma(\mathfrak{X}, M_{\mathfrak{X}})$ is a log p -basis if the induced \mathfrak{Y} -morphism $\mathfrak{X} \rightarrow \mathfrak{Y} \times_{\mathfrak{V}} \widehat{A}_{\mathbb{N}^r}$ is log p -étale, (concerning $\widehat{A}_{\mathbb{N}^r}$, see the notation of 3.1.1).
2. We say that f is log p -smooth if, étale locally on \mathfrak{X} and \mathfrak{Y} , f has a log p -basis. When $\mathfrak{Y} = \mathcal{S}$, we say that \mathfrak{X} is a log p -smooth fine log-formal \mathcal{S} -scheme.

3.1.12. Let \mathfrak{X} be a log p -smooth fine log-formal \mathcal{S} -schemes. Let $\Delta_{\mathfrak{X}/\mathcal{S}}: \mathfrak{X} \hookrightarrow \mathfrak{X} \times_{\mathcal{S}} \mathfrak{X}$ be the diagonal immersion and $\Delta_{\mathfrak{X}/\mathcal{S},(m)}^n := P_{(m)}^n(\Delta_{\mathfrak{X}/\mathcal{S}})$. We denote by $M_{\mathfrak{X}/\mathcal{S},(m)}^n$ the log structure of $\Delta_{\mathfrak{X}/\mathcal{S},(m)}^n$. By denoting abusively by $\Delta_{\mathfrak{X}/\mathcal{S},(m)}^n$ the target of the arrow $\Delta_{\mathfrak{X}/\mathcal{S},(m)}^n$, the underlying morphism of schemes of $\Delta_{\mathfrak{X}/\mathcal{S},(m)}^n \rightarrow \mathfrak{X}$ is finite. We denote by $\mathcal{P}_{\mathfrak{X}/\mathcal{S},(m)}^n$ the coherent $\mathcal{O}_{\mathfrak{X}}$ -algebra corresponding to the underlying formal \mathcal{V} -scheme of $\Delta_{\mathfrak{X}/\mathcal{S},(m)}^n$. Hence, $\Delta_{\mathfrak{X}/\mathcal{S},(m)}^n$ is an exact closed immersion of the form $\Delta_{\mathfrak{X}/\mathcal{S},(m)}^n: \mathfrak{X} \hookrightarrow (\mathrm{Spf} \mathcal{P}_{\mathfrak{X}/\mathcal{S},(m)}^n, M_{\mathfrak{X}/\mathcal{S},(m)}^n)$.

Let $p_1^n, p_0^n: \Delta_{\mathfrak{X}/\mathcal{S},(m)}^n \rightarrow \mathfrak{X}$ be respectively the composition of the canonical morphism $\Delta_{\mathfrak{X}/\mathcal{S},(m)}^n \rightarrow \mathfrak{X} \times_{\mathcal{S}} \mathfrak{X}$ with the right and left projection $\mathfrak{X} \times_{\mathcal{S}} \mathfrak{X} \rightarrow \mathfrak{X}$. From 2.1.1, we check that $p_1^n, p_0^n: \Delta_{\mathfrak{X}/\mathcal{S},(m)}^n \rightarrow \mathfrak{X}$ are strict.

As in paragraph 2.2.1, we can define $\mathcal{D}_{\mathfrak{X}/\mathcal{S}}^{(m)}$, the sheaf of differential operator on \mathfrak{X} of level m .

3.1.13 (Local description). Suppose in this paragraph that $\mathfrak{X} \rightarrow \mathcal{S}$ is endowed with a log p -basis $(b_\lambda)_{\lambda=1,\dots,r}$. Then as in 2.1.2 we get the following local description of $\Delta_{\mathfrak{X}/\mathcal{S},(m)}^n$. We remark first that $(p_1^*(b_\lambda))_{\lambda=1,\dots,r}$ is a log p -basis of p_0 (indeed, the log p -étaleness is stable under base change). Putting $\eta_{\lambda(m)} := \mu_{(m)}^n(b_\lambda) - 1$ (or simply η_λ), where $\mu_{(m)}^n(a)$ is the unique section of $\ker(\mathcal{O}_{\Delta_{\mathfrak{X}/\mathcal{S},(m)}^n}^* \rightarrow \mathcal{O}_{\mathfrak{X}}^*)$ such that we get in $M_{\mathfrak{X}/\mathcal{S},(m)}^n$ the equality $p_1^{n*}(a) = p_0^{n*}(a)\mu_{(m)}^n(a)$, we get the isomorphism of m -PD- $\mathcal{O}_{\mathfrak{X}}$ -algebras

$$\begin{aligned} \mathcal{O}_{\mathfrak{X}} \langle T_1, \dots, T_r \rangle_{(m),n} &\xrightarrow{\sim} \mathcal{P}_{\mathfrak{X}/\mathcal{S},(m)}^n \\ T_\lambda &\mapsto \eta_\lambda, \end{aligned} \quad (3.1.13.1)$$

where the first term is defined as in 1.2.17. In particular, the elements $\{\underline{\eta}^{\{\underline{k}\}_{(m)}}\}_{|\underline{k}| \leq n}$ form an $\mathcal{O}_{\mathfrak{X}}$ -basis of $\mathcal{P}_{\mathfrak{X}/\mathcal{S},(m)}^n$. The corresponding dual basis of $\mathcal{D}_{\mathfrak{X}/\mathcal{S},n}^{(m)}$ will be denoted by $\{\underline{\partial}^{\langle \underline{k} \rangle_{(m)}}\}_{|\underline{k}| \leq n}$. Let $\epsilon_1, \dots, \epsilon_r$ be the canonical basis of \mathbb{N}^r , i.e. the coordinates of ϵ_i are 0 except for the i th term which is 1. We put $\partial_i := \underline{\partial}^{\langle \epsilon_i \rangle_{(m)}}$. We can define the logarithmic transposition as in 2.4.1 and we can check that the properties analogous to the subsection 2.4 are still satisfied in the formal context.

3.2 Sheaf of differential operators of finite level

3.2.1. Let \mathfrak{X} be a log p -smooth fine log-formal \mathcal{S} -schemes. We denote by $\widehat{\mathcal{D}}_{\mathfrak{X}/\mathcal{S}}^{(m)}$ the p -adic completion of $\mathcal{D}_{\mathfrak{X}/\mathcal{S}}^{(m)}$. As in the paragraph 2.2.5, we check that $\mathcal{D}_{X_i/S_i}^{(m)} \xrightarrow{\sim} \mathcal{D}_{\mathfrak{X}/\mathcal{S}}^{(m)} \otimes_{\mathcal{V}} \mathcal{V}/\pi^{i+1}$. Hence $\widehat{\mathcal{D}}_{\mathfrak{X}/\mathcal{S}}^{(m)} \xrightarrow{\sim} \varprojlim_i \mathcal{D}_{X_i/S_i}^{(m)}$.

3.2.2. Let \mathfrak{X} be a log p -smooth fine log-formal \mathcal{S} -schemes. We put $\mathcal{D}_{\mathfrak{X}/\mathcal{S}}^\dagger := \varinjlim_m \widehat{\mathcal{D}}_{\mathfrak{X}/\mathcal{S}}^{(m)}$. This is the “sheaf of differential operators of finite level of \mathfrak{X}/\mathcal{S} ”. When $X \rightarrow S$ is endowed with a log p -basis $(b_\lambda)_{\lambda=1,\dots,n}$ and $\underline{\mathfrak{X}}$ is noetherian, we get the usual description ([Ber96, 2.4.4]): an operator P of $\Gamma(\mathfrak{X}, \mathcal{D}_{\mathfrak{X}/\mathcal{S}}^\dagger)$ is of the form

$$P = \sum_{\underline{k} \in \mathbb{N}^n} a_{\underline{k}} \underline{\partial}^{[\underline{k}]}$$

where $a_{\underline{k}} \in \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ satisfy the condition : there exist some constants $c, \eta \in \mathbb{R}$, with $\eta < 1$, such that for any \underline{k} we have

$$\|a_{\underline{k}}\| \leq c \eta^{|\underline{k}|},$$

where $\|\cdot\|$ is the p -adic norm i.e. whose basis of open neighbourhoods of 0 is given by $(p^n \mathcal{O}_{\mathfrak{X}})_{n \in \mathbb{N}}$.

3.2.3. Let \mathfrak{X} be a log p -smooth fine log-formal \mathcal{S} -scheme. We suppose that the underlying formal \mathcal{V} -scheme of \mathfrak{X} is noetherian. As in [Ber96, 2.2.5], we check that if \mathcal{U} is Zariski open of \mathfrak{X} having a log p -basis then $\Gamma(\mathcal{U}, \mathcal{D}_{X_i/S_i}^{(m)})$ is right and left noetherian. As in [Ber96, 3.1.2], we check that the sheaf $\mathcal{D}_{X_i/S_i}^{(m)}$ is coherent on the right and on the left. As in [Ber96, 3.3.4], this yields that $\widehat{\mathcal{D}}_{\mathfrak{X}/\mathcal{S}}^{(m)}$ is coherent on the right and on the left. As in [Ber96, 3.4.2], this implies that $\widehat{\mathcal{D}}_{\mathfrak{X}/\mathcal{S},\mathbb{Q}}^{(m)}$ is coherent on the right and on the left. By following the proof of [Ber96, 3.5.3], we prove that the extension $\widehat{\mathcal{D}}_{\mathfrak{X}/\mathcal{S},\mathbb{Q}}^{(m)} \rightarrow \widehat{\mathcal{D}}_{\mathfrak{X}/\mathcal{S},\mathbb{Q}}^{(m+1)}$ is flat on the right and on the left. Hence, taking the inductive limits, we obtain the coherence on the right and on the left of $\mathcal{D}_{\mathfrak{X}/\mathcal{S}}^\dagger$. Similarly, we have theorem of type *B* as in [Ber96, 3]: if $\underline{\mathfrak{X}}$ is affine, for any integer $q \geq 1$ we have the vanishing $H^q(\mathfrak{X}, \mathcal{D}_{X_i/S_i}^{(m)}) = 0$, $H^q(\mathfrak{X}, \widehat{\mathcal{D}}_{\mathfrak{X}/\mathcal{S}}^{(m)}) = 0$, $H^q(\mathfrak{X}, \widehat{\mathcal{D}}_{\mathfrak{X}/\mathcal{S},\mathbb{Q}}^{(m)}) = 0$, $H^q(\mathfrak{X}, \mathcal{D}_{\mathfrak{X}/\mathcal{S}}^\dagger) = 0$.

3.2.4 (Frobenius descent). Let \mathfrak{X} be a p -smooth fine log-formal \mathcal{S} -scheme (in particular \mathfrak{X}/\mathcal{S} is strict) so that \mathfrak{X} is noetherian. Using 1.2.12, we remark that the results concerning Frobenius of [Ber00] (e.g. the fact that the functor F_X^* induces an equivalence of categories between coherent $\widehat{\mathcal{D}}_{\mathfrak{X}/\mathcal{S}}^{(m)}$ -modules and coherent $\widehat{\mathcal{D}}_{\mathfrak{X}/\mathcal{S}}^{(m+1)}$ -modules) are still valid (indeed, we have only to copy word by word the arguments or computations) in the context of a p -smooth fine log-formal \mathcal{S} -scheme. Recall that when \mathfrak{X}/\mathcal{S} is not strict, these equivalence of categories involving Frobenius are false in general (see [Mon02]).

As in [Ber00], since $\mathcal{D}_{X_i/\mathcal{S}_i}^{(0)}$ has finite cohomological dimension (the proof is standard), this implies (by completion and Frobenius descent) that so are $\widehat{\mathcal{D}}_{\mathfrak{X}/\mathcal{S}}^{(m)}$ and then by passing to the limit $\mathcal{D}_{\mathfrak{X}/\mathcal{S},\mathbb{Q}}^\dagger$. Since $\mathcal{D}_{\mathfrak{X}/\mathcal{S},\mathbb{Q}}^\dagger$ is also coherent, this yields that the derived category of perfect complexes of left $\mathcal{D}_{\mathfrak{X}/\mathcal{S},\mathbb{Q}}^\dagger$ -modules and that of bounded coherent complexes of left $\mathcal{D}_{\mathfrak{X}/\mathcal{S},\mathbb{Q}}^\dagger$ -modules are the same.

3.2.5 (Overconvergent isocrystals). Let \mathfrak{X} be a log p -smooth fine log-formal \mathcal{S} -schemes and Z be a Cartier divisor of \underline{X}_0 . As in [Ber96, 4.2.3] and with its notation, we define the commutative \mathcal{O}_{X_i} -algebra $\mathcal{B}_{X_i}^{(m)}(Z)$ endowed with a compatible structure of left $\mathcal{D}_{X_i/\mathcal{S}_i}^{(m)}$ -module (see the definition 2.3.4) such that $\mathcal{B}_{X_i}^{(m)}(Z) \rightarrow \mathcal{B}_{X_i}^{(m+1)}(Z)$ is $\mathcal{D}_{X_i/\mathcal{S}_i}^{(m)}$ -linear. We get a structure of $\widehat{\mathcal{D}}_{\mathfrak{X}/\mathcal{S}}^{(m)}$ -module on $\widehat{\mathcal{B}}_{\mathfrak{X}}^{(m)}(Z) = \varprojlim \mathcal{B}_{X_i}^{(m)}(Z)$. From 2.3.5, we get the $\mathcal{O}_{\mathfrak{X}}$ -algebra $\widehat{\mathcal{B}}_{\mathfrak{X}}^{(m)}(Z) \widehat{\otimes} \widehat{\mathcal{D}}_{\mathfrak{X}/\mathcal{S}}^{(m)}$ such that the canonical map $\widehat{\mathcal{D}}_{\mathfrak{X}/\mathcal{S}}^{(m)} \rightarrow \widehat{\mathcal{B}}_{\mathfrak{X}}^{(m)}(Z) \widehat{\otimes} \widehat{\mathcal{D}}_{\mathfrak{X}/\mathcal{S}}^{(m)}$ is a morphism of $\mathcal{O}_{\mathfrak{X}}$ -algebras. We set $\mathcal{O}_{\mathfrak{X}}(\dagger Z) := \varinjlim_m \widehat{\mathcal{B}}_{\mathfrak{X}}^{(m)}(Z)$ and $\mathcal{D}_{\mathfrak{X}/\mathcal{S}}^\dagger(\dagger Z) := \varinjlim_m \widehat{\mathcal{B}}_{\mathfrak{X}}^{(m)}(Z) \widehat{\otimes} \widehat{\mathcal{D}}_{\mathfrak{X}/\mathcal{S}}^{(m)}$. We define an isocrystal on \mathfrak{X} overconvergent along Z to be a coherent $\mathcal{D}_{\mathfrak{X}/\mathcal{S}}^\dagger(\dagger Z)_{\mathbb{Q}}$ -module which is also $\mathcal{O}_{\mathfrak{X}}(\dagger Z)_{\mathbb{Q}}$ -coherent (for the structure induced by the canonical morphism $\mathcal{O}_{\mathfrak{X}}(\dagger Z)_{\mathbb{Q}} \rightarrow \mathcal{D}_{\mathfrak{X}/\mathcal{S}}^\dagger(\dagger Z)_{\mathbb{Q}}$).

3.3 Structure of right $\mathcal{D}_{\mathfrak{X}/\mathcal{S}}^\dagger$ -module on $\omega_{\mathfrak{X}/\mathcal{S}}$

Proposition 3.3.1 (Structure of right $\mathcal{D}_{X/\mathcal{S}}^{(0)}$ -module on $\omega_{X/\mathcal{S}}$). *Let S be a fine log scheme over $\mathbb{Z}/p^{i+1}\mathbb{Z}$ and let (I_S, J_S, γ) be a quasi-coherent m -PD-ideal of \mathcal{O}_S . Let $f: X \rightarrow S$ be a log p -smooth morphism of fine log-schemes such that γ extends to X .*

We have a canonical structure of right $\mathcal{D}_{X/\mathcal{S}}^{(0)}$ -module on $\omega_{X/\mathcal{S}}$ (see notation 2.1.6). Locally, this structure is characterized by the following description. Suppose that $X \rightarrow S$ is endowed with a log p -basis $(b_i)_{i=1,\dots,n}$. Let $d\log b_i$ denotes the image of η_i in $\Gamma(X, \Omega_{X/\mathcal{S}}^1)$. The action of $P \in \mathcal{D}_{X/\mathcal{S}}^{(0)}$ on the section $\text{adlog } b_1 \wedge \dots \wedge d\log b_n$, where a is section of \mathcal{O}_X is given by the formula

$$(a \text{ dlog } b_1 \wedge \dots \wedge d\log b_n) \cdot P = \tilde{P}(a) \text{ dlog } b_1 \wedge \dots \wedge d\log b_n. \quad (3.3.1.1)$$

Proof. 0) It is sufficient to check the independence of the formula 3.3.1.1 with respect to the chosen log p -basis. Suppose that $X \rightarrow S$ is endowed with two log p -bases $(b_i)_{i=1,\dots,n}$ and $(b'_i)_{i=1,\dots,n}$.

1) Let $A = (a_{ij}) \in M_n(\mathcal{O}_X)$ (resp. $A' = (a'_{ij}) \in M_n(\mathcal{O}_X)$) be the matrix such that $\begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix} = A \begin{pmatrix} \partial'_1 \\ \vdots \\ \partial'_n \end{pmatrix}$ (resp. $\begin{pmatrix} \partial'_1 \\ \vdots \\ \partial'_n \end{pmatrix} = A' \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix}$). Hence, we get $A' = A^{-1}$, $\begin{pmatrix} d\log b'_1 \\ \vdots \\ d\log b'_n \end{pmatrix} = {}^t A \begin{pmatrix} d\log b_1 \\ \vdots \\ d\log b_n \end{pmatrix}$ and then $d\log b_1 \wedge \dots \wedge d\log b_n = |A'| d\log b'_1 \wedge \dots \wedge d\log b'_n$. We compute $\partial_{i'} \partial_i = \sum_{j=1}^n \partial_{i'} a_{ij} \partial'_j = \sum_{j=1}^n a_{ij} \partial_{i'} \partial'_j + \sum_{j=1}^n \partial_{i'}(a_{ij}) \partial'_j = \sum_{j,j'=1}^n a_{ij} a_{i'j'} \partial'_{j'} \partial'_j + \sum_{j=1}^n \partial_{i'}(a_{ij}) \partial'_j$. Since $\partial_{i'} \partial_i = \partial_i \partial_{i'}$ and $\partial'_j \partial'_j = \partial'_j \partial'_j$, exchanging i with i' yields $\sum_{j=1}^n \partial_{i'}(a_{ij}) \partial'_j = \sum_{j=1}^n \partial_i(a_{i'j}) \partial'_j$. Hence, for any i, i', j , we have $\partial_{i'}(a_{ij}) = \partial_i(a_{i'j})$ and then (by symmetry) $\partial'_{i'}(a'_{ij}) = \partial'_i(a'_{i'j})$.

2) By symmetry and \mathcal{O}_X -linearity, it is sufficient to check that both actions of ∂_1 on $d\log b_1 \wedge \dots \wedge d\log b_n$ coincides. With the first choice, this is straightforward that we get 0. Now, we consider the action ∂_1 on $d\log b_1 \wedge \dots \wedge d\log b_n$ for the second choice of log p -basis. Since $\partial_1 = \sum_{j=1}^n a_{1j} \partial'_j$, we get $d\log b_1 \wedge \dots \wedge d\log b_n \cdot \partial_1 = (|A'| d\log b'_1 \wedge \dots \wedge d\log b'_n) \cdot \partial_1 = - \sum_{j=1}^n \partial'_j(a_{1j}|A'|) d\log b'_1 \wedge \dots \wedge d\log b'_n$. Hence, we have to check $\sum_{j=1}^n \partial'_j(a_{1j}|A'|) = 0$.

a) We compute $a_{1j}|A'| = \sum_{\sigma \in S_n, \sigma(1)=j} (-1)^{\epsilon(\sigma)} \prod_{i=2}^n a'_{\sigma(i)i}$. Indeed, let L'_1, \dots, L'_n be the rows of A' . We remark that $a_{1j}|A'|$ is equal to the determinant of the matrix A' whose row L'_j is replaced by $a_{1j}L'_j$ and then by $\sum_{l=1}^n a_{1l}L'_l$. Since $AA' = I_n$, we get $\sum_{l=1}^n a_{1l}L'_l = (1, 0, \dots, 0)$. This yields the desired formula.

b) We have $\sum_{j=1}^n \partial'_j(a_{1j}|A'|) = \sum_{\sigma \in S_n, l \in \{2, \dots, n\}} (-1)^{\epsilon(\sigma)} \partial'_{\sigma(1)}(a'_{\sigma(l)l}) \prod_{i=2, i \neq l}^n a'_{\sigma(i)i}$. Indeed, this is a consequence of the formula $\partial'_{\sigma(1)}(\prod_{i=2}^n a'_{\sigma(i)i}) = \sum_{l=2}^n \partial'_{\sigma(1)}(a'_{\sigma(l)l}) \prod_{i=2, i \neq l}^n a'_{\sigma(i)i}$, and of that of part a).

c) We define on $S_n \times \{2, \dots, n\}$ the following equivalence relation. Let (σ, l) and (σ', l') be two elements of $S_n \times \{2, \dots, n\}$. They are equivalent if either $(\sigma', l') = (\sigma, l)$ or $(\sigma', l') = (\sigma \circ (1, l), l)$. From part 1) of the proof, for any (σ, l) , $(\sigma', l') = (\sigma \circ (1, l), l)$ elements of $S_n \times \{2, \dots, n\}$, we get $(-1)^{\epsilon(\sigma')} \partial'_{\sigma'(1)}(a'_{\sigma'(l)l}) \prod_{i=2, i \neq l}^n a'_{\sigma'(i)i} + (-1)^{\epsilon(\sigma)} \partial'_{\sigma(1)}(a'_{\sigma(l)l}) \prod_{i=2, i \neq l}^n a'_{\sigma(i)i} = 0$. \square

Proposition 3.3.2. *Let \mathfrak{X} be a log p -smooth fine log-formal \mathcal{S} -scheme. We suppose that \mathfrak{X} is noetherian and has no p -torsion. We put $\omega_{\mathfrak{X}/\mathcal{S}} := \varprojlim_i \omega_{X_i/S_i}$. There exists a canonical structure of right $\mathcal{D}_{\mathfrak{X}/\mathcal{S}}^\dagger$ -module on $\omega_{\mathfrak{X}/\mathcal{S}}$. It is characterized by the following local formula: suppose that \mathfrak{X} is endowed with a log p -basis $(b_\lambda)_{\lambda=1, \dots, n}$. Let $d\log b_\lambda$ be the image of η_λ in $\Omega_{\mathfrak{X}/\mathcal{S}}^1$. Then, for any integer m , for any differential operator $P \in \mathcal{D}_{\mathfrak{X}/\mathcal{S}}^{(m)}$ and $a \in \mathcal{O}_{\mathfrak{X}}$ we have*

$$(a \, d\log b_1 \wedge \dots \wedge d\log b_n) \cdot P := \tilde{P}(a) \, d\log b_1 \wedge \dots \wedge d\log b_n. \quad (3.3.2.1)$$

Proof. Using 3.3.1, we get a canonical structure of right $\widehat{\mathcal{D}}_{\mathfrak{X}}^{(0)}$ -module on $\omega_{\mathfrak{X}/\mathcal{S}} = \varprojlim_i \omega_{X_i/S_i}$. Hence, we get a structure of right $\widehat{\mathcal{D}}_{\mathfrak{X}, \mathbb{Q}}^{(0)}$ -module on $\omega_{\mathfrak{X}/\mathcal{S}, \mathbb{Q}}$. Since $\mathcal{D}_{\mathfrak{X}/\mathcal{S}}^{(m)} \subset \widehat{\mathcal{D}}_{\mathfrak{X}, \mathbb{Q}}^{(0)}$, we get a structure of right $\mathcal{D}_{\mathfrak{X}/\mathcal{S}}^{(m)}$ -module on $\omega_{\mathfrak{X}/\mathcal{S}, \mathbb{Q}}$. Let us check that $\omega_{\mathfrak{X}/\mathcal{S}}$ is a sub $\mathcal{D}_{\mathfrak{X}/\mathcal{S}}^{(m)}$ -module of $\omega_{\mathfrak{X}/\mathcal{S}, \mathbb{Q}}$. Since this is local, we can suppose that \mathfrak{X} is endowed with a log p -basis $(b_\lambda)_{\lambda=1, \dots, n}$. We compute that the right $\mathcal{D}_{\mathfrak{X}/\mathcal{S}}^{(m)}$ -action on $\omega_{\mathfrak{X}/\mathcal{S}, \mathbb{Q}}$ is given by the formula 3.3.2.1. This implies that $\omega_{\mathfrak{X}/\mathcal{S}}$ is a sub $\mathcal{D}_{\mathfrak{X}/\mathcal{S}}^{(m)}$ -module of $\omega_{\mathfrak{X}/\mathcal{S}, \mathbb{Q}}$. Using (a right log version of) [Ber90, 3.1.3], this yields that $\omega_{\mathfrak{X}/\mathcal{S}}$ is endowed with a canonical structure of $\widehat{\mathcal{D}}_{\mathfrak{X}}^{(m)}$ -module. Since these structures are compatible with $\widehat{\mathcal{D}}_{\mathfrak{X}}^{(m)} \rightarrow \widehat{\mathcal{D}}_{\mathfrak{X}}^{(m+1)}$, we are done. \square

Corollary 3.3.3. *Let \mathfrak{X} be a log p -smooth fine log-formal \mathcal{S} -scheme such that \mathfrak{X} is noetherian and has no p -torsion. Let i be an integer. There exists a canonical structure of right $\mathcal{D}_{X_i/S_i}^{(m)}$ -module on ω_{X_i/S_i} . It is characterized by the following local formula: suppose that \mathfrak{X} is endowed with a log p -basis $(b_\lambda)_{\lambda=1, \dots, n}$. Let $d\log b_\lambda$ be the image of η_λ in Ω_{X_i/S_i}^1 . Then, for any integer m , for any differential operator $P \in \mathcal{D}_{X_i/S_i}^{(m)}$ and $a \in \mathcal{O}_{X_i}$ we have*

$$(a \, d\log b_1 \wedge \dots \wedge d\log b_n) \cdot P := \tilde{P}(a) \, d\log b_1 \wedge \dots \wedge d\log b_n. \quad (3.3.3.1)$$

Proof. This is a consequence of 3.3.2. \square

Corollary 3.3.4. *Let \mathfrak{X} be a log p -smooth fine log-formal \mathcal{S} -scheme such that \mathfrak{X} is noetherian and has no p -torsion. The functor $- \otimes_{\mathcal{O}_{X_i}} \omega_{X_i}$ (resp. $- \otimes_{\mathcal{O}_{X_i}} \omega_{\mathfrak{X}}$) is an equivalence of categories between that of left $\mathcal{D}_{X_i/S_i}^{(m)}$ -modules (resp. left $\mathcal{D}_{\mathfrak{X}/\mathcal{S}}^\dagger$ -modules) and that of right $\mathcal{D}_{X_i/S_i}^{(m)}$ -modules (resp. right $\mathcal{D}_{\mathfrak{X}/\mathcal{S}}^\dagger$ -modules). The functor $- \otimes_{\mathcal{O}_{X_i}} \omega_{X_i}^{-1}$ (resp. $- \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega_{\mathfrak{X}}^{-1}$) is a quasi-inverse functor. Both functors preserve the coherence. These functors are the "twisted structure" of \mathcal{D} -module.*

3.4 Pushforwards, extraordinary pull-backs and duality

Let $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism of log p -smooth fine log-formal \mathcal{S} -schemes. We suppose that the formal \mathcal{V} -schemes \mathfrak{X} and \mathfrak{Y} are noetherian and have no p -torsion. We can follow Berthelot's construction of pushforwards, extraordinary pull-backs and duality as explained in [Ber02] or [Ber00]. For the reader, let's describe a little the construction.

3.4.1. By functoriality, the left $\mathcal{D}_{X_i/S_i}^{(m)}$ -module $f^* \mathcal{D}_{Y_i/S_i}^{(m)}$ is in fact endowed with a structure of $(\mathcal{D}_{X_i/S_i}^{(m)}, f^{-1} \mathcal{D}_{Y_i/S_i}^{(m)})$ -bimodule which we will denote by $\mathcal{D}_{X_i \rightarrow Y_i/S_i}^{(m)}$. By twisting (see 3.3.4), we get a $(f^{-1} \mathcal{D}_{Y_i/S_i}^{(m)}, \mathcal{D}_{X_i/S_i}^{(m)})$ -bimodule by setting $\mathcal{D}_{Y_i \leftarrow X_i/S_i}^{(m)} := \omega_{X_i/S_i} \otimes_{\mathcal{O}_{X_i}} f_g^* (\mathcal{D}_{Y_i/S_i}^{(m)} \otimes_{\mathcal{O}_{Y_i}} \omega_{Y_i/S_i}^{-1})$, where the index g means that we choose the left structure of the left $\mathcal{D}_{Y_i/S_i}^{(m)}$ -bimodule $\mathcal{D}_{Y_i/S_i}^{(m)} \otimes_{\mathcal{O}_{Y_i}} \omega_{Y_i/S_i}^{-1}$. Next, we put $\widehat{\mathcal{D}}_{\mathfrak{X} \rightarrow \mathfrak{Y}/\mathcal{S}}^{(m)} := \varprojlim_i \mathcal{D}_{X_i \rightarrow Y_i/S_i}^{(m)}$ and $\widehat{\mathcal{D}}_{\mathfrak{Y} \leftarrow \mathfrak{X}/\mathcal{S}}^{(m)} := \varprojlim_i \mathcal{D}_{Y_i \leftarrow X_i/S_i}^{(m)}$. Finally, $\mathcal{D}_{\mathfrak{X} \rightarrow \mathfrak{Y}/\mathcal{S}}^\dagger := \varinjlim_m \widehat{\mathcal{D}}_{\mathfrak{X} \rightarrow \mathfrak{Y}/\mathcal{S}}^{(m)}$ and $\mathcal{D}_{\mathfrak{Y} \leftarrow \mathfrak{X}/\mathcal{S}}^\dagger := \varinjlim_m \widehat{\mathcal{D}}_{\mathfrak{Y} \leftarrow \mathfrak{X}/\mathcal{S}}^{(m)}$. Let $d_{\mathfrak{X}}$ (resp. $d_{\mathfrak{Y}}$) be the Krull dimension of \mathfrak{X} (resp. \mathfrak{Y}). We denote by $D^b(\mathcal{D}_{\mathfrak{X}/\mathcal{S}, \mathbb{Q}}^\dagger)$, (resp. $D_{\text{coh}}^b(\mathcal{D}_{\mathfrak{X}/\mathcal{S}, \mathbb{Q}}^\dagger)$, resp. $D_{\text{parf}}(\mathcal{D}_{\mathfrak{X}/\mathcal{S}, \mathbb{Q}}^\dagger)$) the derived category of bounded (resp. and bounded and coherent, resp. perfect) complexes of left $\mathcal{D}_{\mathfrak{X}/\mathcal{S}, \mathbb{Q}}^\dagger$ -modules.

1. As in [Ber02, 4.3.2.2], we get the functor $f^!: D_{\text{coh}}^b(\mathcal{D}_{\mathfrak{Y}/\mathcal{S}, \mathbb{Q}}^\dagger) \rightarrow D^b(\mathcal{D}_{\mathfrak{X}/\mathcal{S}, \mathbb{Q}}^\dagger)$ by setting, for any object \mathcal{F} of $D_{\text{coh}}^b(\mathcal{D}_{\mathfrak{Y}/\mathcal{S}, \mathbb{Q}}^\dagger)$,

$$f^!(\mathcal{F}) := \mathcal{D}_{\mathfrak{X} \rightarrow \mathfrak{Y}/\mathcal{S}, \mathbb{Q}}^\dagger \otimes_{f^{-1} \mathcal{D}_{\mathfrak{Y}/\mathcal{S}, \mathbb{Q}}^\dagger}^{\mathbb{L}} f^{-1} \mathcal{F} [d_{\mathfrak{X}} - d_{\mathfrak{Y}}].$$

2. As in [Ber02, 4.3.7.1], we get the functor $f_+ : D_{\text{coh}}^b(\mathcal{D}_{\mathfrak{X}/S, \mathbb{Q}}^\dagger) \rightarrow D^b(\mathcal{D}_{Y/S, \mathbb{Q}}^\dagger)$ by setting, for any object \mathcal{E} of $D_{\text{coh}}^b(\mathcal{D}_{\mathfrak{X}/S, \mathbb{Q}}^\dagger)$,

$$f_+(\mathcal{E}) := \mathbb{R}f_*(\mathcal{D}_{Y \leftarrow \mathfrak{X}/S, \mathbb{Q}}^\dagger \otimes_{\mathcal{D}_{\mathfrak{X}/S, \mathbb{Q}}^\dagger}^{\mathbb{L}} \mathcal{E}).$$

3. As in [Ber02, 4.3.10], we get the functor $\mathbb{D}_{\mathfrak{X}} : D_{\text{parf}}(\mathcal{D}_{\mathfrak{X}/S, \mathbb{Q}}^\dagger) \rightarrow D_{\text{parf}}(\mathcal{D}_{\mathfrak{X}/S, \mathbb{Q}}^\dagger)$ by posing, for any object \mathcal{E} of $D_{\text{parf}}(\mathcal{D}_{\mathfrak{X}/S, \mathbb{Q}}^\dagger)$,

$$\mathbb{D}_{\mathfrak{X}}(\mathcal{E}) := \mathbb{R}\text{Hom}_{\mathcal{D}_{\mathfrak{X}/S, \mathbb{Q}}^\dagger}(\mathcal{E}, \mathcal{D}_{\mathfrak{X}/S, \mathbb{Q}}^\dagger \otimes_{\mathcal{O}_{\mathfrak{X}, \mathbb{Q}}} \omega_{\mathfrak{X}/S, \mathbb{Q}}^{-1})[d_{\mathfrak{X}}].$$

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